

Introduction to Electricity & Magnetism

with Applications to Optics & Electronics

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Part I

Theory

Chapter 1

Teaser

There is a force like gravity but a million trillion trillion trillion times stronger. We experience it as the solidity of matter, the colors of light, the deflection of a compass needle, and the plasma discharge upon touching a doorknob on a dry day.

1.1 Gravity

Gravitational "charge" or mass $M > 0$ has SI unit of **kilogram**,

$$\text{unit}[M] = \text{kg}. \tag{1.1}$$

Place two point (or spherical) masses M a distance r apart. Observe a **gravitational force** proportional to the first mass times the second mass diluted by the area of the separation sphere,

$$F_{\mathcal{G}} \propto M\frac{M}{4\pi r^2} \tag{1.2}$$

or

$$F_{\mathcal{G}} = G\frac{M^2}{r^2}. \tag{1.3}$$

In SI units, the gravitational constant

$$G = 6.67\times 10^{-11}\frac{\text{N m}^2}{\text{kg}^2} = 66.7\frac{\text{pN m}^2}{\text{kg}^2}, \tag{1.4}$$

so that two 1kg masses separated by 1m attract each other with the minuscule force of 66.7pN $\approx$ 100pN.

Gravity binds planets to stars and apples to Earth according to Newton's laws. However, ordinary matter consists of atoms, each containing a swarm of electrons moving at nearly light speed and bound to within a nanometer of a nucleus of protons and neutrons. What is this binding force? Gravity is far too weak, but there is another force, the electric force, and another charge, the electric charge.

1.2 Electricity

Unlike mass, which is always positive, electric charge can be positive or negative. Furthermore, electric charges manifest different forces depending on whether they are in relative motion or rest. Results of such experiments can be combined in startling ways.

Use Scotch tape to demonstrate two kinds of electrical charge. Tape two pieces together back-to-front and quickly unpeel them. They should attract one another. Similarly prepare a second pair of tape pieces and bring them near the others to observe both attraction *and* repulsion; that's something gravity doesn't do. (Don't underestimate Scotch tape: in a Nobel-Prize-winning experiment, physicists used Scotch tape to extract flakes of graphene from a chunk of graphite; in another experiment, physicists took x-ray photographs of their fingers by unpeeling Scotch tape in a vacuum.)

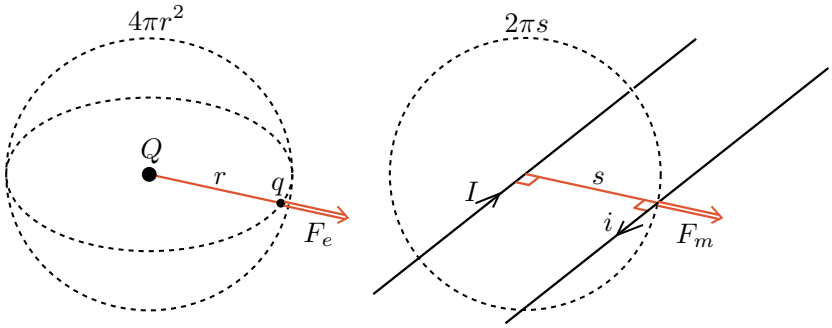


Figure 1.1: Spherical symmetry of the electric force between stationary charges (left) and circular symmetry of the magnetic force between moving charges (right).

Electrical charge $Q \lessgtr 0$ has SI unit of **coulomb**,

$$\text{unit}[Q] = \text{C}. \tag{1.5}$$

In a **Coulomb experiment**, transfer an excess charge Q to two spheres a radial distance r apart. Observe an **electric force** proportional to the first charge times the second charge diluted by the surface area of the Fig. 1.1 separation sphere,

$$F_{\mathcal{E}} \propto Q\frac{Q}{4\pi r^2} \tag{1.6}$$

or

$$\epsilon_0 F_{\mathcal{E}} = \frac{Q^2}{4\pi r^2}. \tag{1.7}$$

In natural units, the electric constant ϵ_0 is unity, but in SI units

$$\epsilon_0 = 8.85 \times 10^{-12}\frac{\text{C}^2}{\text{N m}^2} = 8.85\frac{\text{C}^2}{\text{TN m}^2}. \tag{1.8}$$

Alternately, in analogy with the Eq. 1.3 gravitational force, write

$$F_{\mathcal{E}} = k\frac{Q^2}{r^2}, \tag{1.9}$$

where the common combination

$$k = \frac{1}{4\pi\epsilon_0} = 8.99\times 10^9 \frac{\text{N m}^2}{\text{C}^2} = 8.99\frac{\text{GN m}^2}{\text{C}^2}, \tag{1.10}$$

so that two 1C charges separated by 1m attract each other with the enormous force of 8.99GN $\approx$ 10GN

Consider two electrons of charge $Q_e = -1.60\times 10^{-19}$C and mass $m_e = 9.11\times 10^{-31}$kg a distance r apart. The ratio of their electric repulsion to their gravitational attraction is

$$\begin{aligned}\frac{F_{\mathcal{E}}}{F_{\mathcal{G}}} &= \frac{k\,Q_e^2/r^2}{G\,m_e^2/r^2} = \frac{k\,Q_e^2}{G\,m_e^2}\\ &= \frac{\left(8.99\times 10^9\,\frac{\text{N m}^2}{\text{C}^2}\right)\left(1.60\times 10^{-19}\text{C}\right)^2}{\left(6.67\times 10^{-11}\,\frac{\text{N m}^2}{\text{kg}^2}\right)\left(9.11\times 10^{-31}\text{kg}\right)^2}\\ &\approx \frac{10^{+10}10^{-38}}{10^{-10}10^{-60}} = 10^{42},\end{aligned} \tag{1.11}$$

independent of distance. Hence, the electric repulsion

$$F_{\mathcal{E}} \approx 10^{42}F_{\mathcal{G}} = 10^6 10^{12}10^{12}10^{12}F_{\mathcal{G}} \tag{1.12}$$

is about a million trillion trillion trillion times the gravitational attraction.

If electric charge dQ passes through a surface in time dt, a current

$$I = \frac{dQ}{dt} \tag{1.13}$$

flows. Electrical current $I \lessgtr 0$ has SI unit of **ampere**,

$$\text{unit}[I] = \frac{\text{C}}{\text{s}} = \text{A}. \tag{1.14}$$

In an **Ampère experiment**, separate parallel currents I by a perpendicular distance $s = r_\perp$. Observe a **magnetic force** per unit length proportional to the first current times the second current diluted by the circumference of the Fig. 1.1 separation circle,

$$\frac{F_{\mathcal{B}}}{\ell} \propto I\frac{I}{2\pi s} \tag{1.15}$$

or

$$\mu_0^{-1}\frac{F_{\mathcal{B}}}{\ell} = \frac{I^2}{2\pi s}. \tag{1.16}$$

In natural units, the magnetic constant μ_0 is unity, but in SI units

$$\mu_0 = 1.26 \times 10^{-6} \frac{\mathrm{N\ s^2}}{\mathrm{C^2}} = 1.26 \frac{\mu\mathrm{N\ s^2}}{\mathrm{C^2}} = 1.26 \frac{\mu\mathrm{N}}{\mathrm{A^2}}. \tag{1.17}$$

The Ampère and Coulomb experiments have unexpected implications. The inverse square root of the *product* of the electric and magnetic constants is the speed

$$\begin{aligned} c &= \frac{1}{\sqrt{\epsilon_0 \mu_0}} \\ &= \frac{1}{\sqrt{\left(8.85 \times 10^{-12} \frac{\mathrm{C^2}}{\mathrm{N\ m^2}}\right)\left(1.26 \times 10^{-6} \frac{\mathrm{N\ s^2}}{\mathrm{C^2}}\right)}} \\ &= 3.00 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}} = 0.300 \frac{\mathrm{m}}{\mathrm{ns}}, \end{aligned} \tag{1.18}$$

or about one foot per nanosecond, or one billion kilometers per hour, or seven times around Earth in one second. This is the speed of light – but speed relative to what? It's not *relative* to anything; it's the unique *invariant* speed. (Hence the symbol c for "constant".) It's the also the speed limit of the universe, and it can be found by experiments on stationary charges and steady currents.

There's more. The inverse square root of the *quotient* of the electric and magnetic constants is the resistance

$$\begin{aligned} Z_0 &= \sqrt{\frac{\mu_0}{\epsilon_0}} \\ &= \sqrt{\frac{1.26 \times 10^{-6} \frac{\mathrm{N\ s^2}}{\mathrm{C^2}}}{8.85 \times 10^{-12} \frac{\mathrm{C^2}}{\mathrm{N\ m^2}}}} \\ &= 377 \frac{\mathrm{N\ m\ s}}{\mathrm{C^2}} = 377 \frac{\mathrm{J/C}}{\mathrm{C/s}} = 377 \frac{\mathrm{V}}{\mathrm{A}} = 377 \Omega. \end{aligned} \tag{1.19}$$

This is known as the "impedance of free space". The **irradiance** or **intensity** of an electromagnet wave in a vacuum is the average power per unit area

$$\mathcal{I} = \frac{\mathcal{E}_{\mathrm{rms}}^2}{Z_0}, \tag{1.20}$$

where $\mathcal{E}_{\mathrm{rms}}$ is the root mean square electric field and

$$\mathrm{unit}[\mathcal{I}] = \frac{\mathrm{W}}{\mathrm{m^2}}. \tag{1.21}$$

An object with impedance Z_0 will not reflect electromagnetic radiation. Such an airplane would be invisible to radar.

1.3 Problems

1. You want to steal the moon (properly known as "Luna" in Latin and other languages).

   (a) What electrical charge in TC must you add to Earth and Luna to cancel their gravitational attraction? (Hint: Derive a formula for the charge, find and substitute data, check units and significant figures.)

   (b) How many kg of electrons will accomplish this? Wow!

2. You transfer one percent of your electrons to the person sitting next to you in class.

   (a) What attractive force in N is between you and your neighbor (both approximated as spheres of water)?

   (b) Compare this force to the "weight of Earth" (taken to be Earth's mass times the magnitude of the gravitational field at Earth's surface, $g = 9.81\text{N/kg} = 9.81\text{m/s}^2$). Wow!

Chapter 2

Coulomb

Coulomb's law relates electric charge to electric force.

2.1 Electric Charge

Electric charges combine like positive and negative numbers, and the two "flavors" of charge are conventionally called "positive" and "negative". By contrast, the "color" charges of the strong nuclear interaction combine like additive color mixing, and the three flavors are conventionally called "red", "green", and "blue".

Think of the two flavors of electric charge as opposite manifestations of one quality: positive and negative are two kinds of charges in the way that left and right are two kinds of handedness, or before and after are two kinds of temporal ordering. The **CPT theorem** states that physical laws are invariant under transformations involving simultaneous inversions of charge, parity (handedness), and time. Thus, for every elementary particle, there exists a kind of mirror-image particle with all of its reversible properties reversed, including its electric charge. Corresponding to the negative electron e^- is the positive anti-electron (positron) e^+; corresponding to the positive proton p^+ is the negative antiproton p^-.

The process of **pair annihilation**

$$e^+ + e^- \to 2\gamma \tag{2.1}$$

converts matter and antimatter into flashes of light, where the Greek letter γ denotes an energetic "gamma ray" **photon**. The inverse process of **pair production**

$$\gamma + N \to e^+ + e^- + N \tag{2.2}$$

can create antimatter, where N is an atomic nucleus that "catalyzes" the process by absorbing some momentum, and the gamma ray's energy $E \geq (2m_e)c^2 \approx$

1.0MeV = 0.16pJ. Lightning in terrestrial thunderstorms accelerates electrons in the process of **bremsstrhalung**

$$e^- + N \to e^- + N + \gamma \tag{2.3}$$

occasionally radiating gamma rays, some of which subsequently produce antimatter beams via the Eq. 2.2 pair production.

In the visible universe today there are about one billion photons (mostly in the **Cosmic Microwave Background**) for every **baryon** (mostly protons and neutrons). This suggests a slight one-part-in-a-billion asymmetry in the matter-antimatter processes of the very early universe.

Electric charge is **quantized**, $Q = nQ_e$, where n is an integer. (However, permanently bound quarks in protons and neutrons are assigned fractional charges of $\pm Q_e/3$.) In every process, electric charge is **conserved**. For example, the total electric charge is zero both before and after the Eq. 2.1-2.2 pair annihilation and production. For all observers, electric charge is **invariant**. Unlike measurements of space and time, all observers agree on measurements of charge regardless of their motion.

2.2 Electric Force

The vector electric force on a stationary point charge q displaced $\vec{\boldsymbol{\imath}}$ from a stationary charge q' is

$$\epsilon_0 \vec{F} = +q \frac{q'}{4\pi \boldsymbol{\imath}^2} \hat{\boldsymbol{\imath}}, \tag{2.4}$$

where $\boldsymbol{\imath}$ is the magnitude of the separation and $\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{\imath}}/\boldsymbol{\imath}$ is its direction.

The denominator $4\pi\boldsymbol{\imath}^2$ in the Eq. 2.4 Coloumb's law reflects the fact that at a distance $\boldsymbol{\imath}$ the influence of the charge q' has spread over and is diluted by a sphere of surface area $4\pi\boldsymbol{\imath}^2$. The direction $\hat{\boldsymbol{\imath}}$ reflects the fact that the line joining the charges selects a unique direction in an otherwise isotropic space. However, if the charges are moving, a deflecting "magnetic" force supplements the Coulomb force. If the charges are accelerating, the emission of electromagnetic radiation further modifies the Coulomb force.

The Eq. 1.11 calculation demonstrates the enormous strength of the electric force compared to the gravitational force. Gravity controls the large-scale structure of the universe not due to its strength but due to its single mindedness: it's always attractive and never repulsive. The almost perfect intermingling of positive and negative charge in ordinary matter nearly neutralizes the electric force. However, try to pass your fist through a wall. Despite the apparent solidity, hand and wall are both mainly empty space, because atoms themselves, with their relatively massive positively charged nuclei and their relatively light negatively charged electron distributions, are nearly empty space. Contact slightly distorts the electron distributions, as in Fig. 2.1, resulting in volume exclusion and the "contact forces" of introductory physics.

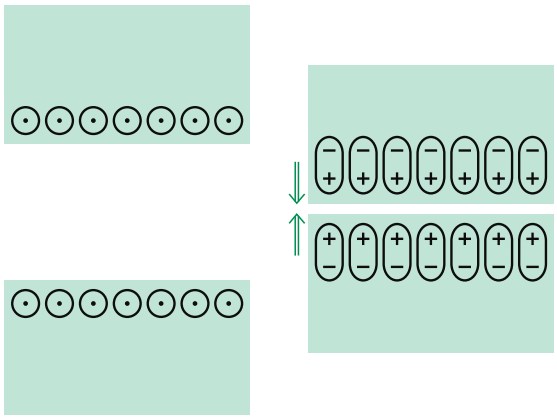


Figure 2.1: Pushing together two solids distorts atomic electron distributions, separating the atomic charge centers; the ++ repulsion is stronger than the +− attraction.

2.3 Electric Force Components

Locate charges relative to the Fig. 2.2 rectangular coordinate system. Let $\hat{x}, \hat{y}, \hat{z}$ be unit vectors in the directions of increasing x, y, z. (Alternate notations for these basis vectors include $\hat{x}_1, \hat{x}_2, \hat{x}_3$ or $\hat{e}_x, \hat{e}_y, \hat{e}_z$ or $\mathbf{i}, \mathbf{j}, \mathbf{k}$.) Relative to the origin $\mathcal{O}$, a charge q is at

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x\hat{x} + y\hat{y} + z\hat{z}. \tag{2.5}$$

A second charge q' is at

$$\vec{r}\,' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = x' \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y' \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z' \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x'\hat{x} + y'\hat{y} + z'\hat{z}. \tag{2.6}$$

The relative displacement of the charges

$$\vec{\boldsymbol{\imath}} = \vec{r} - \vec{r}\,' = \begin{bmatrix} x - x' \\ y - y' \\ z - z' \end{bmatrix} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z} \tag{2.7}$$

has magnitude

$$\boldsymbol{\imath} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} > 0 \tag{2.8}$$

and direction

$$\hat{\boldsymbol{\imath}} = \frac{\vec{\boldsymbol{\imath}}}{\boldsymbol{\imath}} = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \begin{bmatrix} x - x' \\ y - y' \\ z - z' \end{bmatrix}. \tag{2.9}$$

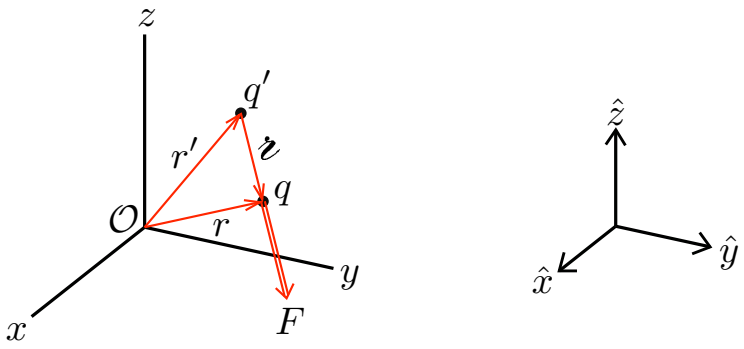


Figure 2.2: Locating charges q and q' with respect to rectangular coordinates $\{x, y, z\}$ and basis vectors $\{\hat{x}, \hat{y}, \hat{z}\}$.

Read the notation $\hat{\boldsymbol{r}} = \vec{\boldsymbol{r}}/\boldsymbol{r}$ as "script r hat equals script r vector over script r". The "hat" or circumflex symbol represents an arrow head, which gives direction information without length information (while the arrow symbol includes the shaft and so gives both).

Using this notation, the Eq. 2.4 electric force

$$\epsilon_0 \vec{F} = q\frac{q'}{4\pi \boldsymbol{r}^2}\hat{\boldsymbol{r}} = \frac{qq'}{4\pi \boldsymbol{r}^3}\vec{\boldsymbol{r}} \tag{2.10}$$

expands to

$$\epsilon_0 \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \frac{qq'}{4\pi\Big((x-x')^2+(y-y')^2+(z-z')^2\Big)^{3/2}} \begin{bmatrix} x-x' \\ y-y' \\ z-z' \end{bmatrix} \tag{2.11}$$

or the three scalar equations

$$\epsilon_0 F_x = \frac{qq'}{4\pi}\frac{x-x'}{\big((x-x')^2+(y-y')^2+(z-z')^2\big)^{3/2}}, \tag{2.12a}$$

$$\epsilon_0 F_y = \frac{qq'}{4\pi}\frac{y-y'}{\big((x-x')^2+(y-y')^2+(z-z')^2\big)^{3/2}}, \tag{2.12b}$$

$$\epsilon_0 F_z = \frac{qq'}{4\pi}\frac{z-z'}{\big((x-x')^2+(y-y')^2+(z-z')^2\big)^{3/2}}. \tag{2.12c}$$

2.4 Superposition

Implicit in the vector nature of Coulomb's law is the **superposition** principle: electric forces add vectorially,

$$\vec{F} = \vec{F}_1 + \vec{F}_2. \tag{2.13}$$

More generally, if a charge q experiences a force $\vec{F}_n$ due to charge Q_n alone, then q experiences a force

$$\vec{F} = \sum_{n=1}^{N} \vec{F}_n \tag{2.14}$$

due to the discrete charge distribution

$$Q = \sum_{n=1}^{N} Q_n. \tag{2.15}$$

Similarly, if a charge q experiences an infinitesimal force $d\vec{F}$ due to and infinitesimal charge dQ alone, then q experiences a force

$$\vec{F} = \int d\vec{F} \tag{2.16}$$

due to the continuous charge distribution

$$Q = \int dQ. \tag{2.17}$$

This simple and empirical combination law is profoundly important – don't take if for granted! Furthermore, since the contact forces of everyday life are electrical in nature, they too obey the superposition principle.

2.4.1 Example: Dipole Bisector

An **electric dipole** consists of a charge $+Q > 0$ a distance $d = 2r$ from a charge $-Q$. Consider a test charge $q > 0$ a separation s along the dipole's perpendicular bisector, as in Fig. 2.3.

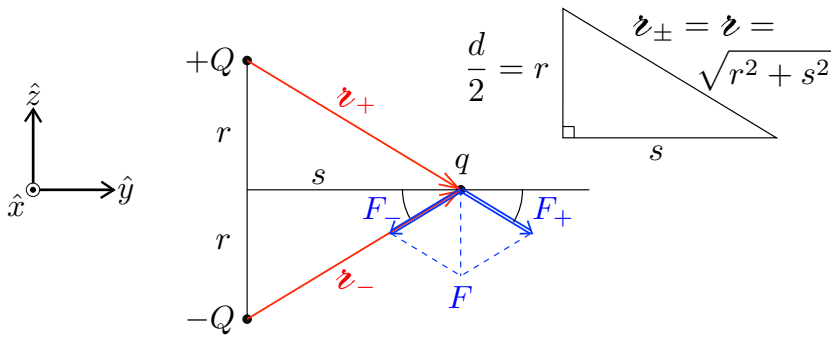


Figure 2.3: Force on charge q on the perpendicular bisector of a dipole configuration of the charges $\pm Q$.

The force on charge q is the superposition

$$\begin{aligned}
\epsilon_0 \vec{F}^{\perp} &= \epsilon_0 \vec{F}^{\perp}_{+} + \epsilon_0 \vec{F}^{\perp}_{-} \\
&= q\frac{Q_+}{4\pi \boldsymbol{\imath}_+^2}\hat{\boldsymbol{\imath}}_+ + q\frac{Q_-}{4\pi \boldsymbol{\imath}_-^2}\hat{\boldsymbol{\imath}}_- \\
&= q\frac{Q_+}{4\pi \boldsymbol{\imath}_+^3}\vec{\boldsymbol{\imath}}_+ + q\frac{Q_-}{4\pi \boldsymbol{\imath}_-^3}\vec{\boldsymbol{\imath}}_- \\
&= q\frac{Q}{4\pi \boldsymbol{\imath}^3}\begin{bmatrix} 0 \\ +s \\ -r \end{bmatrix} - q\frac{Q}{4\pi \boldsymbol{\imath}^3}\begin{bmatrix} 0 \\ +s \\ +r \end{bmatrix} \\
&= q\frac{Q}{4\pi \boldsymbol{\imath}^3}\begin{bmatrix} 0 \\ 0 \\ -2r \end{bmatrix} \\
&= -q\frac{Q}{4\pi \boldsymbol{\imath}^3}2r\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= -\frac{qQr}{2\pi(r^2+s^2)^{3/2}}\hat{z}.
\end{aligned} \tag{2.18}$$

Hence, the only nonzero component is

$$\epsilon_0 F_z^{\perp} = -\frac{qQ}{2\pi}\frac{r}{(r^2+s^2)^{3/2}} < 0, \tag{2.19}$$

and the force is *parallel* to the line defined by the dipole. Check some limiting cases: if q is midway between $\pm Q$, then

$$\epsilon_0 F_z^{\perp}[s=0] = -\frac{qQ}{2\pi}\frac{r}{(r^2+0^2)^{3/2}} = -2\,q\frac{Q}{4\pi r^2}, \tag{2.20}$$

which is clearly correct, as both dipole charges are equidistant from the test charge and are forcing it down; if q is far away, then

$$\epsilon_0 F_z^{\perp}[s \gg \ell] \approx -\frac{qQ}{2\pi}\frac{r}{(0^2+s^2)^{3/2}} = -\frac{qQr}{2\pi s^3} \propto \frac{1}{s^3}, \tag{2.21}$$

and the force decreases as the inverse *cube* of the distance! Although each dipole charge contributes an inverse square force, their forces tend to cancel because they are of opposite sign. ■

2.4.2 Example: Discrete Charges

Organize the computation more formally if the superposition is more complicated. For example, suppose charges

$$Q = q, \quad Q_1 = -3q, \quad Q_2 = 2q, \quad Q_3 = q, \tag{2.22}$$

are at positions

$$\vec{r} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} d, \quad \vec{r}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} d, \quad \vec{r}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} d, \quad \vec{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d, \tag{2.23}$$

as in Fig. 2.4.

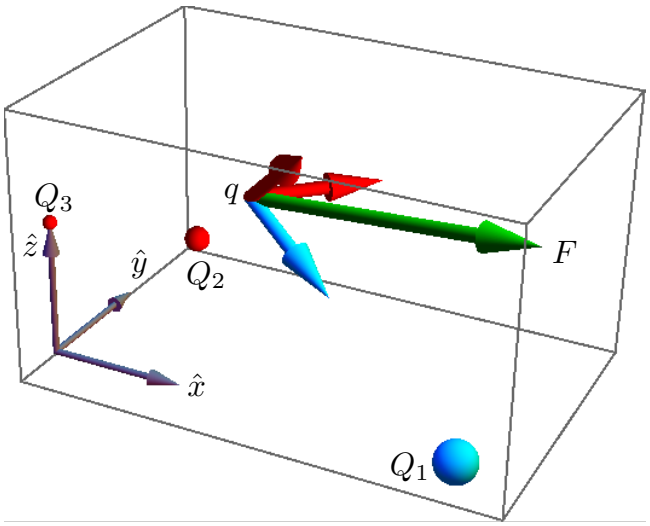


Figure 2.4: Force on charge q due to a three-dimensional configuration of charges Q_n.

The relative displacements $\vec{\boldsymbol{\imath}}_n = \vec{r} - \vec{r}_n$ are

$$\vec{\boldsymbol{\imath}}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} d, \quad \vec{\boldsymbol{\imath}}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} d, \quad \vec{\boldsymbol{\imath}}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} d, \tag{2.24}$$

with magnitudes

$$\boldsymbol{\imath}_1 = \sqrt{6}d, \quad \boldsymbol{\imath}_2 = \sqrt{3}d, \quad \boldsymbol{\imath}_3 = \sqrt{2}d, \tag{2.25}$$

and directions

$$\hat{\boldsymbol{\imath}}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \hat{\boldsymbol{\imath}}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \hat{\boldsymbol{\imath}}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{2.26}$$

The force on charge q is the superposition

$$\begin{aligned}
\epsilon_0\vec{F} &= \epsilon_0\vec{F}_1 + \epsilon_0\vec{F}_2 + \epsilon_0\vec{F}_3 \\
&= Q\frac{Q_1}{4\pi\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + Q\frac{Q_2}{4\pi\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + Q\frac{Q_3}{4\pi\mathscr{r}_3^3}\hat{\boldsymbol{\mathscr{r}}}_3 \\
&= \left(-\frac{1}{2\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix} + \frac{2}{3\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} + \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}\right)\frac{q^2}{4\pi d^2} \\
&= \frac{1}{6\sqrt{6}}\begin{bmatrix}+6+4\sqrt{2}+3\sqrt{3}\\-3-4\sqrt{2}+3\sqrt{3}\\-3+4\sqrt{2}\end{bmatrix}\frac{q^2}{4\pi d^2} \\
&\approx \begin{bmatrix}1.1\\-0.24\\0.18\end{bmatrix}\frac{q^2}{4\pi d^2}. \qquad (2.27)
\end{aligned}$$

■

2.5 Electric Field

What is the *mechanism* of the electric force? What are the "gears" and "wheels" that make it work? Is it an "instantaneous action-at-a-distance"?

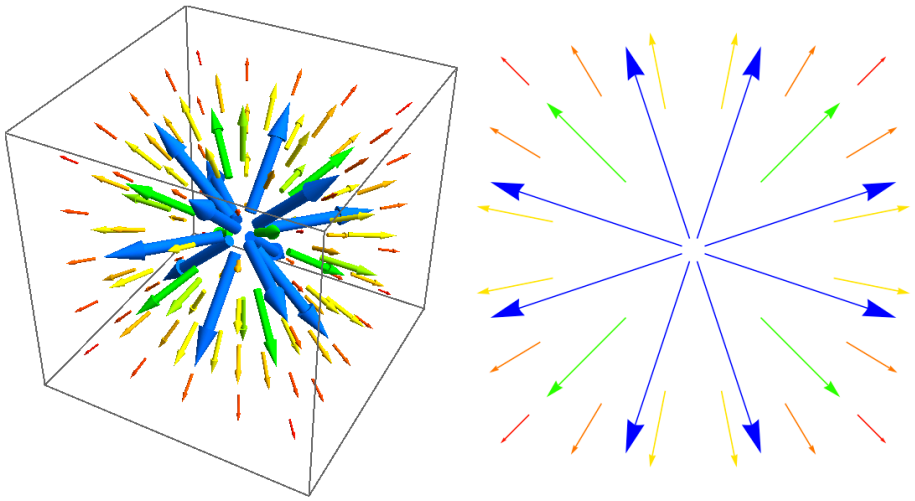


Figure 2.5: Arrows visualize a monopole electric field $\vec{\mathcal{E}}$ in perspective and cross section.

Experience has shown that the **field paradigm** is a better way to describe

the electric force. Decompose the Eq. 1.7 Coulomb's law into two parts by introducing the **electric field** $\vec{\mathcal{E}}$ via

$$\vec{F}_{\mathcal{E}} = q\vec{\mathcal{E}}, \tag{2.28}$$

where

$$\epsilon_0 \vec{\mathcal{E}} = \frac{Q}{4\pi \boldsymbol{\imath}^2} \hat{\boldsymbol{\imath}}. \tag{2.29}$$

This replaces the action-at-a-distance paradigm of

$$Q \Leftrightarrow q \tag{2.30}$$

with the field paradigm of

$$Q \Leftrightarrow \vec{\mathcal{E}} \Leftrightarrow q. \tag{2.31}$$

Shake a charge Q. According to Coulomb's law, the force on charge q changes, but this doesn't happen instantaneously. Instead, a disturbance in the electric field emanates from Q and travels to q at light speed c. Such **electromagnetic waves** carry energy and momentum and in that sense are as real as atoms.

The electric field of a point charge

$$\epsilon_0 \vec{\mathcal{E}}[\vec{r}] = \epsilon_0 \frac{\vec{F}_{\mathcal{E}}}{q} = \frac{Q}{4\pi r^2} \hat{r} \tag{2.32}$$

or

$$\vec{\mathcal{E}}[\vec{r}] = \frac{\vec{F}_{\mathcal{E}}}{q} = k\frac{Q}{r^2} \hat{r} \tag{2.33}$$

is the electric force per unit charge, with SI unit of

$$\text{unit}[\mathcal{E}] = \frac{\text{N}}{\text{C}}. \tag{2.34}$$

Analogously, the gravitational field of Earth

$$\vec{g}[\vec{r}] = \frac{\vec{F}_{\mathcal{G}}}{m} = -G\frac{M_\oplus}{r^2} \hat{r} \tag{2.35}$$

is the gravitational force per unit mass, with SI unit of

$$\text{unit}[g] = \frac{\text{N}}{\text{kg}}, \tag{2.36}$$

where the familiar $g[R_\oplus] = GM_\oplus/R_\oplus^2 = 9.81\text{N/kg} = 9.81\text{m/s}^2$.

Since the electric field is the electric force per unit charge, the superposition principle extends from the force to the field. Thus, if $\vec{\mathcal{E}}_n$ is the electric field due to a charge Q_n alone, then

$$\vec{\mathcal{E}} = \sum_{n=1}^{N} \vec{\mathcal{E}}_n \tag{2.37}$$

is the electric field due to the discrete charge distribution

$$Q = \sum_{n=1}^{N} Q_n. \tag{2.38}$$

Similarly, if $d\vec{\mathcal{E}}$ is the infinitesimal electric field due to an infinitesimal charge dQ alone, then

$$\vec{\mathcal{E}} = \int d\vec{\mathcal{E}} \tag{2.39}$$

is the electric field due to the continuous charge distribution

$$Q = \int dQ. \tag{2.40}$$

Visualize an electric field using arrows or field lines. The magnitudes and directions of the arrows at selected points indicate the magnitude and direction of the electric field $\vec{\mathcal{E}}[\vec{r}]$ at those points, as in Figs. 2.5-2.6, where the arrows are colored according to their lengths. However, often the range of electric field magnitudes is so large that it is impractical to draw the arrow lengths proportional to the electric field magnitudes. The arrows can overlap, and the space point to which they apply can be unclear.

Field lines are everywhere tangent to the electric field, and the density of lines are proportional to the electric field magnitudes, as in Fig. 2.7. Electric field lines start on positive charges and stop on negative charges, and they never cross (else the electric field would have two different directions at the intersection point). Imagine tension parallel to the lines and pressure perpendicular to them. To be quantitatively accurate, draw the field lines in three dimensions. However, as this is difficult, often settle for two-dimensional renderings.

2.6 Electric Dipole

An electric **monopole** consists of a single charge Q. An electric **dipole** consists of charges of equal magnitude but opposite sign $\pm Q$ separated by a distance $d = 2r$ and characterized by the **dipole moment**

$$\vec{\mu}_{\mathcal{E}} = Q\vec{d}, \tag{2.41}$$

which points from the negative to the positive charge, as in Fig. 2.8. An important idealization is the **point dipole** characterized by the limit where the distance d shrinks to zero as the charge Q diverges to infinity, such that the product

$$\vec{\mu}_{\mathcal{E}} = \lim_{\substack{Q\to\infty \\ d\to 0}} Q\vec{d} \tag{2.42}$$

remains finite.

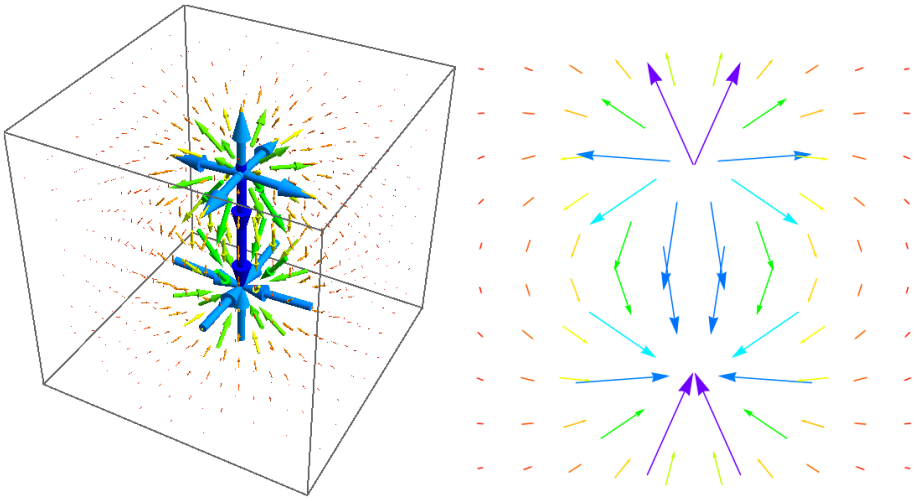


Figure 2.6: Arrows visualize a dipole electric field $\vec{\mathcal{E}}$ in perspective and cross section.

Leveraging the Eq. 2.18 calculation, the electric field in the equatorial plane of the Fig. 2.3 dipole is

$$\epsilon_0 \mathcal{E}_z^\perp = \epsilon_0 \frac{F_z^\perp}{q} = -\frac{Q}{2\pi} \frac{r}{(r^2+s^2)^{3/2}} = -\frac{Q}{2\pi} \frac{d/2}{(d^2/4+s^2)^{3/2}} \tag{2.43}$$

or

$$\epsilon_0 \mathcal{E}_z^\perp = -\frac{\mu\varepsilon}{4\pi(s^2+d^2/4)^{3/2}} \approx -\frac{\mu\varepsilon}{4\pi s^3}, \tag{2.44}$$

for large distances $s \gg d$. The electric field along the axis of the dipole is

$$\begin{aligned} \epsilon_0 \vec{\mathcal{E}}^\parallel &= \epsilon_0 \vec{\mathcal{E}}_+^\parallel + \epsilon_0 \vec{\mathcal{E}}_-^\parallel \\ &= \frac{Q}{4\pi(z-r)^2}\hat{z} - \frac{Q}{4\pi(z+r)^2}\hat{z} \\ &= \frac{Q}{4\pi}\frac{4zr}{(z^2-r^2)^2}\hat{z} \\ &= \frac{Qd}{4\pi}\frac{2z}{(z^2-d^2/4)^2}\hat{z} \end{aligned} \tag{2.45}$$

or

$$\epsilon_0 \mathcal{E}_z^\parallel = \frac{\mu\varepsilon z}{2\pi(z^2-d^2/4)^2} \approx \frac{\mu\varepsilon}{2\pi z^3}, \tag{2.46}$$

for large distances $z \gg d$. Thus, if $\epsilon_0 \vec{\mathcal{E}}_0 = \vec{\mu}_\mathcal{E}/4\pi r^3$, then at large distances $r \gg d$ the dipole's equatorial electric field is $-\vec{\mathcal{E}}_0$ and its axial field is $+2\vec{\mathcal{E}}_0$.

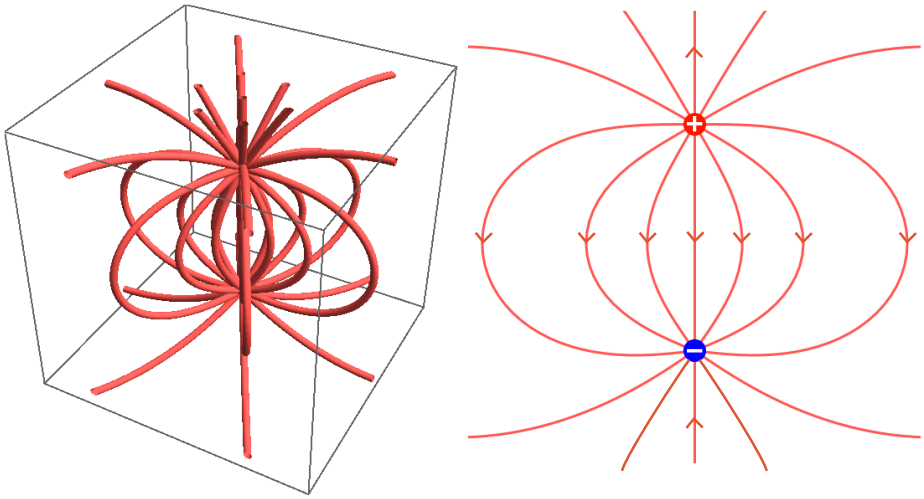


Figure 2.7: Field lines visualize a dipole electric field $\vec{\mathcal{E}}$ in perspective and cross section.

Unlike the monopole field, which is spherically symmetric and decreases like $1/r^2$, the dipole field is spherically asymmetric and decays like $1/r^3$. Summarize the dipole electric field by the single expression

$$\epsilon_0 \vec{\mathcal{E}}[\vec{r}] = \frac{3(\vec{\mu}_\mathcal{E} \cdot \hat{r})\hat{r} - \vec{\mu}_\mathcal{E}}{4\pi r^3}, \tag{2.47}$$

where $\vec{r} = r\hat{r}$ is an arbitrary point in space.

The force on a dipole in a constant electric field is zero,

$$\vec{F} = \vec{F}_+ + \vec{F}_- = +Q\vec{\mathcal{E}} - Q\vec{\mathcal{E}} = \vec{0}, \tag{2.48}$$

as in Fig. 2.8. However, the **moment of force** or **torque** on the dipole about its center is nonzero,

$$\begin{aligned}
\vec{\tau} &= \vec{\tau}_+ + \vec{\tau}_- \\
&= \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- \\
&= \vec{r}_+ \times (+Q\vec{\mathcal{E}}) + \vec{r}_- \times (-Q\vec{\mathcal{E}}) \\
&= Q(\vec{r}_+ - \vec{r}_-) \times \vec{\mathcal{E}} \\
&= Q\vec{d} \times \vec{\mathcal{E}} \\
&= \vec{\mu}_\mathcal{E} \times \vec{\mathcal{E}}.
\end{aligned} \tag{2.49}$$

The torque tends to align the dipole with the electric field.

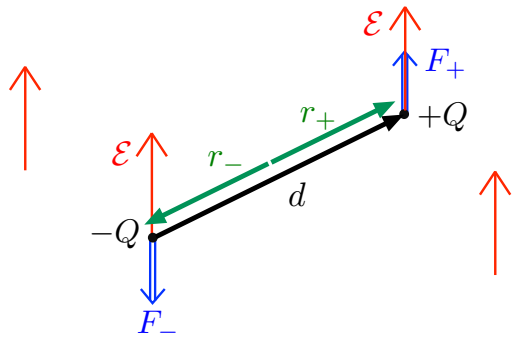


Figure 2.8: Electric dipole $\vec{\mu}_{\mathcal{E}} = Q\vec{d}$ in a constant electric field $\vec{\mathcal{E}}$.

2.7 Continuous Charge Distributions

Electric fields of continuous charge distributions are uncountably infinite superpositions of infinitesimal fields. Compute these uncountable sums using integrals.

2.7.1 Example: Line Charge

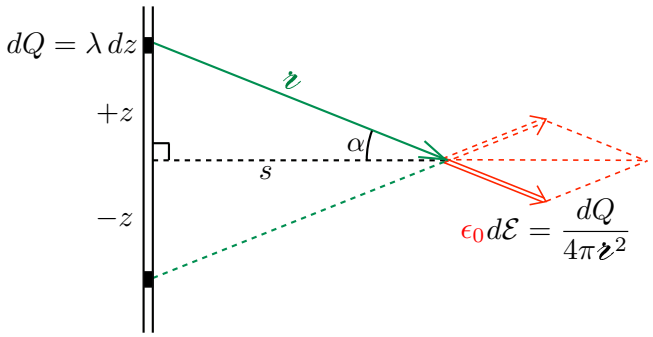


Figure 2.9: A line charge's infinitesimal charge dQ produces an infinitesimal electric field $d\vec{\mathcal{E}}$.

Consider the electric field $\vec{\mathcal{E}}$ a perpendicular distance $s = r_\perp$ from the Fig. 2.9 infinitely long line charge. Assume a linear charge density λ with SI unit of

$$\text{unit}[\lambda] = \frac{\text{C}}{\text{m}}. \tag{2.50}$$

Regard the line as infinitely many infinitesimal charges elements $dQ = \lambda\, dz$. Treat these as point charges. By symmetry, the electric field components parallel

to the line cancel in pairs. The field perpendicular to the line is

$$\mathcal{E}_s = \int d\mathcal{E}_s = \int d\mathcal{E} \cos\alpha \tag{2.51}$$

or

$$\epsilon_0 \mathcal{E}_s = \int \frac{dQ}{4\pi \boldsymbol{\imath}^2} \cos\alpha = \frac{\lambda}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{\boldsymbol{\imath}^2} \cos\alpha. \tag{2.52}$$

Express each of the interrelated variables z, $\boldsymbol{\imath}$, and α in terms of the angle α. From the Fig. 2.9 geometry,

$$\tan\alpha = \frac{z}{s} \tag{2.53}$$

so

$$z = s\tan\alpha, \tag{2.54}$$

where $z = \pm\infty$ corresponds to $\alpha = \pm\pi/2$. Differentiate to find

$$dz = s(\sec\alpha)^2 d\alpha. \tag{2.55}$$

Also from the geometry,

$$\cos\alpha = \frac{s}{\boldsymbol{\imath}} \tag{2.56}$$

so

$$\boldsymbol{\imath} = s\sec\alpha. \tag{2.57}$$

Substituting Eq. 2.55 and Eq. 2.57 into Eq. 2.52, the perpendicular electric field component

$$\begin{aligned}
\epsilon_0 \mathcal{E}_s &= \frac{\lambda}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{s(\sec\alpha)^2 d\alpha}{(s\sec\alpha)^2} \cos\alpha \\
&= \frac{\lambda}{4\pi s} \int_{-\pi/2}^{\pi/2} \cos\alpha \, d\alpha \\
&= \frac{\lambda}{4\pi s} \sin\alpha \Big|_{-\pi/2}^{\pi/2} \\
&= \frac{\lambda}{2\pi s},
\end{aligned} \tag{2.58}$$

and the vector electric field

$$\epsilon_0 \vec{\mathcal{E}} = \frac{\lambda}{2\pi s} \hat{s}, \tag{2.59}$$

which is exact for an infinite line charge and a good approximation near the middle of a long but finite line charge. The $2\pi s$ in the denominator reflects the circular symmetry of the line (just as $4\pi r^2$ reflects the spherical symmetry of a point). ■

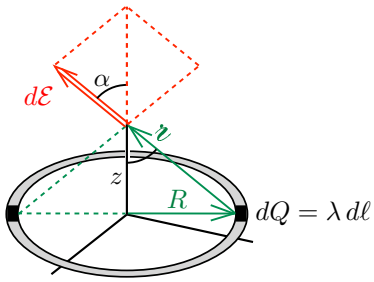


Figure 2.10: A ring charge's infinitesimal charge dQ produces an infinitesimal electric field $d\vec{\mathcal{E}}$.

2.7.2 Example: Ring Charge

Consider the electric field $\vec{\mathcal{E}}$ a distance z along the axis of the Fig. 2.10 ring charge of radius R, linear charge density λ, and total charge $Q = \lambda\, 2\pi R$.

Regard the ring as infinitely many infinitesimal charges elements $dQ = \lambda\, d\ell$. Treat these as point charges. By symmetry, the horizontal electric field components cancel in pairs, leaving the vertical field

$$\mathcal{E}_z = \int d\mathcal{E}_z = \int d\mathcal{E} \cos\alpha \tag{2.60}$$

or

$$\epsilon_0 \mathcal{E}_z = \int \frac{dQ}{4\pi \ell^2} \cos\alpha = \int \frac{\lambda d\ell}{4\pi \ell^2} \cos\alpha. \tag{2.61}$$

Since the distance ℓ and the angle α are constant around the loop,

$$\begin{aligned} \epsilon_0 \mathcal{E}_z &= \frac{\lambda}{4\pi \ell^2} \cos\alpha \int d\ell \\ &= \frac{\lambda}{4\pi \ell^2} \left(\frac{z}{\ell}\right) (2\pi R) \\ &= \frac{\lambda\, 2\pi R\, z}{4\pi \ell^3} \\ &= \frac{Qz}{4\pi (z^2 + R^2)^{3/2}}, \end{aligned} \tag{2.62}$$

and hence the vector electric field on the axis

$$\epsilon_0 \vec{\mathcal{E}} = \frac{Qz}{4\pi (z^2 + R^2)^{3/2}} \hat{z}. \tag{2.63}$$

Check some limiting cases: at the ring's center, $z = 0$ and $\vec{\mathcal{E}} = \vec{0}$, which is clearly correct by symmetry; far from the ring's center, $z \gg R$ and

$$\epsilon_0 \vec{\mathcal{E}}[z \gg R] \approx \frac{Qz}{4\pi (z^2 + 0^2)^{3/2}} \hat{z} = \frac{Q}{4\pi z^2} \hat{z}, \tag{2.64}$$

so the ring acts like a point charge. ■

2.7.3 Example: Plane Charge

Consider the electric field $\vec{\mathcal{E}}$ a distance z above the Fig. 2.11 infinite plane charge.

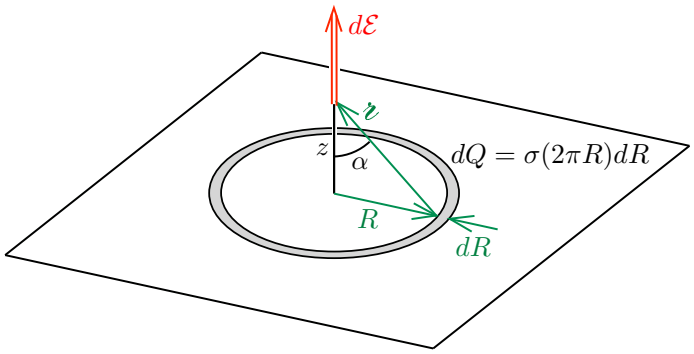


Figure 2.11: A plane charge's infinitesimal ring charge dQ produces an infinitesimal electric field $d\vec{\mathcal{E}}$.

Assume an area charge density σ with SI unit of

$$\text{unit}[\sigma] = \frac{\text{C}}{\text{m}^2}. \tag{2.65}$$

Leveraging the Sec 2.7.2 example, regard the plane as infinitely many infinitesimal ring elements $dQ = \sigma(2\pi R)dR$. The resulting electric field

$$\mathcal{E}_z = \int d\mathcal{E}_z \tag{2.66}$$

or

$$\begin{aligned} \epsilon_0 \mathcal{E}_z &= \int \frac{dQz}{4\pi(z^2+R^2)^{3/2}} \\ &= \int_0^\infty \frac{\sigma(2\pi R)dR\,z}{4\pi(z^2+R^2)^{3/2}} \\ &= \frac{\sigma}{2}\int_0^\infty \frac{R\,dR\,z}{\mathscr{r}^3}. \end{aligned} \tag{2.67}$$

As in the Sec. 3.2.1 example, express each of the interrelated variables R and $\mathscr{r}$ in terms of the angle α, where α ranges from 0 to $\pi/2$ as R ranges from

0 to ∞. From the Fig. 2.11 geometry,

$$\begin{aligned}\epsilon_0\mathcal{E}_z &= \frac{\sigma}{2}\int_0^{\pi/2}\frac{(z\tan\alpha)z(\sec\alpha)^2 d\alpha\, z}{(z\sec\alpha)^3}\\ &= \frac{\sigma}{2}\int_0^{\pi/2}\sin\alpha\, d\alpha\\ &= \frac{\sigma}{2}(-\cos\alpha)\Big|_0^{\pi/2}\\ &= \frac{\sigma}{2},\end{aligned} \tag{2.68}$$

and the vector electric field

$$\epsilon_0\vec{\mathcal{E}} = \frac{\sigma}{2}\hat{z}, \tag{2.69}$$

which is exact for an infinite plane charge and a good approximation near the middle of a large but finite plane charge. The 2 in the denominator reflects the two sides of the plane. ■

Table 2.1 summarizes common charge density symbols, and Table 2.2 summarizes the electric field of point, line, and plane charges. The fields closely follow from the geometry.

Table 2.1: One, two, and three-dimensional charge densities.

Dimension	Symbol	SI Unit
line	λ	$\mathrm{C/m}^1$
area	σ	$\mathrm{C/m}^2$
solid	ρ	$\mathrm{C/m}^3$

Table 2.2: Geometry determines the fields of simple charge distributions.

Charge Distribution	Electric Field
point	$\epsilon_0\mathcal{E}_r = \dfrac{Q}{4\pi r^2}$
line	$\epsilon_0\mathcal{E}_s = \dfrac{\lambda}{2\pi s}$
area	$\epsilon_0\mathcal{E}_\perp = \dfrac{\sigma}{2}$

2.8 Problems

1. Three collinear charges are separated by equal distances. What is the ratio of the first two charges if the total electric force on the third charge is zero?

2. Three charges Q_1, Q_2, and Q_3 are at the vertices of an equilateral triangle of side d. In each case below, find the force $\vec{F}_3$ on Q_3.

   (a) $Q_1 = q$, $Q_2 = -q$, $Q_3 = -q$.

   (b) $Q_1 = q$, $Q_2 = 2q$, $Q_3 = 3q$.

3. If charges

$$Q = q > 0, \quad Q_1 = q, \quad Q_2 = 2q, \quad Q_3 = -2q,$$

   are at positions

$$\vec{r} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} d, \quad \vec{r}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} d, \quad \vec{r}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} d, \quad \vec{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d,$$

   what is the force $\vec{F}$ on q?

4. From the same pivot two vertical strings of length ℓ suspend tiny balls of mass m. Add excess charge Q to each ball so they repel each other to a small horizontal distance d, like leaves in an electroscope.

   (a) Derive a formula for the charge Q. (Hint: Small separation of the balls $d \ll \ell$ suggests a handy approximation.)

   (b) If the mass $m = 0.010\text{kg}$, the length $\ell = 1.2\text{m}$, and the distance $d = 0.050\text{m}$, what is the excess charge Q in nC?

5. Identical charges Q are at the vertices of the following regular polygons of side d. Find the electric field $\vec{\mathcal{E}}$ at their centers. Carefully justify your answers! (Hint: Try arranging the electric field vectors head to tail.)

   (a) Hexagon.

   (b) Heptagon.

6. Charges of $-nQ < 0$ are at the nth hour dots of a 12-hour clock. (For example, at 2 o'clock there is a charge of $-2Q$, and so on.) To what time does the electric field at the clock's center point?

7. A charge $-2Q$ is adjacent to a charge $5Q$. What fraction of the electric field lines escape to infinity? (Hint: Each electric field line starts on positive charge and stops on negative charge.)

8. Using the Appendix B dot product formulas, check the Eq. 2.47 dipole electric field in the following special cases.

   (a) In the plane perpendicular to the dipole.

   (b) Along the axis of the dipole.

9. A point particle of mass m and electric charge $-q < 0$ is on the axis of the Fig. 2.10 ring charge $Q > 0$. For small $z \ll R$ perpendicular displacements from the ring plane, show that the point charge undergoes simple harmonic motion, and derive a formula for its angular frequency ω.

10. Modify the Sec. 3.2.1 example to find the electric field $\vec{\mathcal{E}}$ on the perpendicular bisector of a *finite* line charge of length ℓ and total charge Q.

Chapter 3

Gauss

The first of the four Maxwell equations that summarize electromagnetism, Gauss's law provides a new perspective on Coulomb's law. It relates electric fields on a closed surface to the charges inside. The relationship is simplest in terms of a quantity called **flux**.

3.1 Electric Flux

In hydrodynamics, **volume flux** Φ_V is the rate (in, say, liters per second) that water flows through an area. If the area a is orthogonal to a flow of velocity $\vec{v}$, as on the right in Fig. 3.1, in time dt a volume $dV = (v\,dt)a$ passes through, and the volume flux

$$\Phi_V = \frac{dV}{dt} = va. \tag{3.1}$$

If the area is skewed through an angle α, as in the center in Fig. 3.1, in time dt a volume $dV = (v\,dt)a\cos\alpha$ passes through, and the volume flux

$$\Phi_V = \frac{dV}{dt} = va\cos\alpha = \vec{v}\cdot\vec{a}, \tag{3.2}$$

where the vector $\vec{a}$ is *perpendicular* to the area it represents (as any area defines a unique line perpendicular to it). If the area is curved, as on the right in Fig. 3.1, compute the flux through all infinitesimal sub-areas $d\vec{a}$ and sum to get the total flux

$$\Phi_V = \iint_a \vec{v}\cdot d\vec{a}, \tag{3.3}$$

where the double integral reflects the two-dimensionality of the area. If the area is closed, without boundaries like spherical shell, write

$$\Phi_V = \oiint_a \vec{v}\cdot d\vec{a}. \tag{3.4}$$

A positive flux $\Phi_V > 0$ indicates that the water is diverging (perhaps from a faucet), while a negative flux $\Phi_V < 0$ indicates that the water is converging (perhaps to a drain).

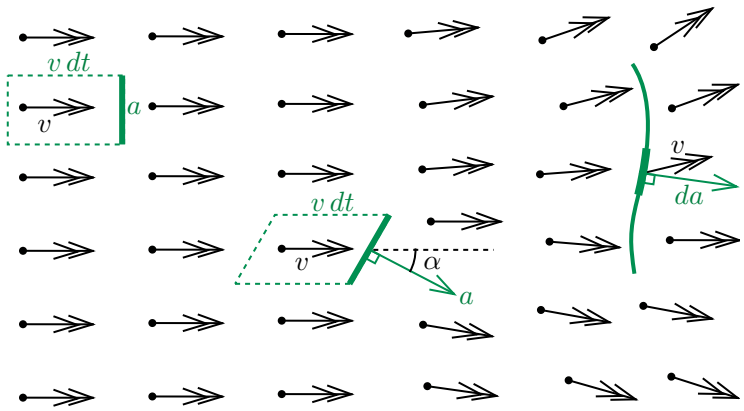


Figure 3.1: Hydrodynamic cross section of water with velocity field $\vec{v}[\vec{r}]$ passing through areas a orthogonal (left), skewed (center), and curved (right).

In analogy with hydrodynamical flux, the **electric flux** $\Phi_\mathcal{E}$ through an open surface a is

$$\Phi_\mathcal{E} = \iint_a \vec{\mathcal{E}} \cdot d\vec{a} \tag{3.5}$$

and the electric flux through a closed surface is

$$\Phi_\mathcal{E} = \oiint_a \vec{\mathcal{E}} \cdot d\vec{a}, \tag{3.6}$$

where the infinitesimal area elements $d\vec{a}$ always point outward. The electric flux is the product of the average perpendicular component of the electric field $\langle \mathcal{E}_\perp \rangle$ and the area a,

$$\Phi_\mathcal{E} = \iint_a \vec{\mathcal{E}} \cdot d\vec{a} = \iint_a \mathcal{E}_\perp da = \left(\frac{1}{a} \iint_a \mathcal{E}_\perp da \right) a = \langle \mathcal{E}_\perp \rangle a. \tag{3.7}$$

3.1.1 Example: Point Charge Flux

Consider the electric flux $\Phi_\mathcal{E}$ through a spherical shell a of radius r due to a point charge Q at its center. Since both the field $\vec{\mathcal{E}}[\vec{r}] = \mathcal{E}_r[r]\hat{r}$ and the infinitesimal area elements $d\vec{a} = da\,\hat{r}$ are radial,

$$\Phi_\mathcal{E} = \oiint_a \vec{\mathcal{E}} \cdot d\vec{a} = \oiint_a \mathcal{E}_r da = \mathcal{E}_r[r] \oiint_{r=\text{const}} da = \epsilon_0^{-1} \frac{Q}{4\pi r^2} 4\pi r^2 = \epsilon_0^{-1} Q \tag{3.8}$$

or

$$\epsilon_0 \Phi_\mathcal{E} = Q. \tag{3.9}$$

The electric flux is proportional to the charge. Indeed, in natural units, where the electric constant ϵ_0 is unity, the electric flux *is* the charge. ■

3.2 Gauss's Law

Generalize the Eq. 3.9 proportionality between flux and charge to an arbitrary closed surface by pairing inner and outer patches, as in Fig. 3.2. The flux through the outer patch is the flux through the inner patch,

$$d\Phi_{\mathcal{E}}[R] = \vec{\mathcal{E}}[R]\cdot d\vec{A} = \epsilon_0^{-1}\frac{Q}{4\pi R^2}dA\cos\alpha = \epsilon_0^{-1}\frac{Q}{4\pi r^2}da = \vec{\mathcal{E}}[r]\cdot d\vec{a} = d\Phi_{\mathcal{E}}[r], \tag{3.10}$$

as $dA\cos\alpha = dA' = daR^2/r^2$. The flux that enters the small patch exits the large patch, as would a stream of photons or bullets. Hence, the flux through the arbitrary surface is the flux through the sphere and both are proportional to the charge enclosed. Generalize this to an arbitrary charge distribution by superposition,

$$\Phi_{\mathcal{E}} = \oiint_a \vec{\mathcal{E}}\cdot d\vec{a} = \oiint_a \sum_n \vec{\mathcal{E}}_n\cdot d\vec{a} = \sum_n \oiint_a \vec{\mathcal{E}}_n\cdot d\vec{a} = \sum_n \epsilon_0^{-1}Q_n = \epsilon_0^{-1}Q, \tag{3.11}$$

which is **Gauss's law**,

$$\epsilon_0\Phi_{\mathcal{E}} = Q_{\text{in}}, \tag{3.12}$$

where the additional subscript "in" is a reminder that Q now represents the total, algebraic sum of the charges in the volume V bounded by the closed surface a.

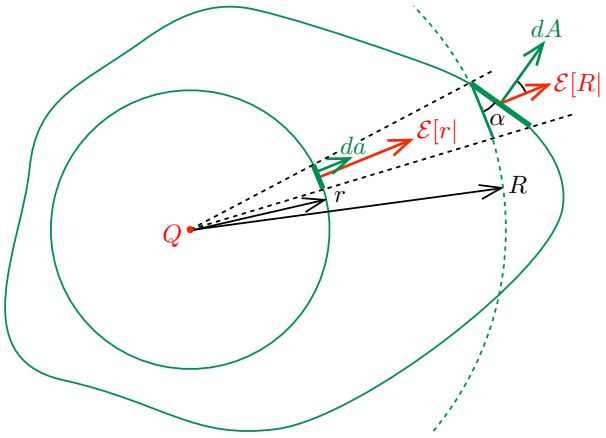


Figure 3.2: Cross sections of a sphere and an arbitrary surface enclosing a charge Q. Flux through the inner patch is the same as the flux through the outer patch.

By analogy, if a *point* mass M has a gravitational field

$$\vec{g} = -\frac{GM}{r^2}\hat{r}, \tag{3.13}$$

and the gravitational flux through a closed area a is

$$\Phi_{\mathcal{G}} = \oiint_a \vec{g} \cdot d\vec{a}, \tag{3.14}$$

then Gauss's law for gravity is

$$\Phi_{\mathcal{G}} = -4\pi GM. \tag{3.15}$$

It follows that the gravitational field of any *spherical* mass is the same as that of a point mass at its center, and an apple falls from a tree as if all of Earth's mass were concentrated at its core. This is one of Newton's great "sphere" theorems.

Gauss's law is always valid, but it is most useful in cases of high symmetry. Put Gauss's Law to work by using it to find the electric fields of symmetric charge distributions.

3.2.1 Example: Line Charge

Consider an infinite line with charge density λ, as in Fig. 3.3.

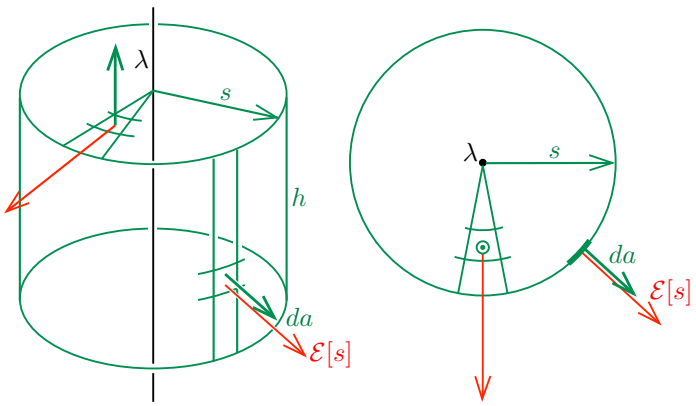


Figure 3.3: Convenient Gaussian surface for an infinite line charge is a concentric cylinder or "can", shown in perspective and from above.

By symmetry, the electric field $\vec{\mathcal{E}}[\vec{r}] = \mathcal{E}[s]\hat{s}$. While Gauss's law works for any surface, a finite concentric cylinder or "can" mirrors the symmetry of the

problem. The electric flux through the surface

$$\begin{aligned}\Phi_{\mathcal{E}} &= \oiint_{\text{can}} \vec{\mathcal{E}} \cdot d\vec{a} \\ &= \iint_{\text{top}} \vec{\mathcal{E}} \cdot d\vec{a} + \iint_{\text{side}} \vec{\mathcal{E}} \cdot d\vec{a} + \iint_{\text{bot}} \vec{\mathcal{E}} \cdot d\vec{a} \\ &= 0 + \mathcal{E}[s] \iint_{\text{side}} da + 0 \\ &= \mathcal{E}[s](2\pi s)h. \end{aligned} \tag{3.16}$$

The total charge enclosed

$$Q = \lambda h. \tag{3.17}$$

Substitute flux and charge into the Eq. 3.12 Gauss's law to get

$$\epsilon_0 \mathcal{E}[s](2\pi s)h = \lambda h \tag{3.18}$$

and conclude

$$\epsilon_0 \mathcal{E}[s] = \frac{\lambda}{2\pi s}, \tag{3.19}$$

which agrees with the Eq. 2.59 electric field obtained by direct integration of Coulomb's law. ■

3.2.2 Example: Plane Charge

Consider and infinite plane with charge density σ, as in Fig. 3.4.

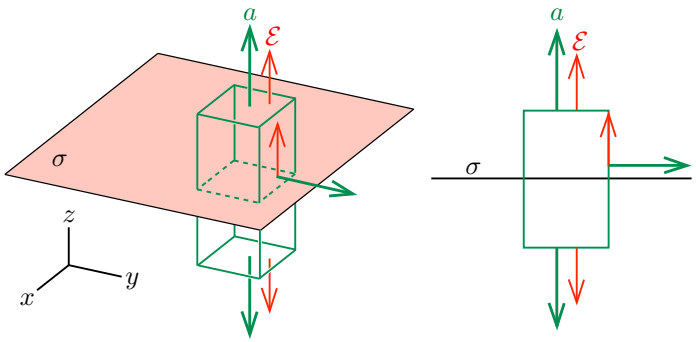


Figure 3.4: Convenient Gaussian surface for an infinite plane charge is a rectangular box straddling the plane, shown in perspective and cross sction.

By symmetry, the electric field $\vec{\mathcal{E}}[\vec{r}] = \mathcal{E}[z]\hat{z}$. While Gauss's law works for any surface, a finite rectangular box straddling the plane mirrors the symmetry

of the problem. The electric flux through the surface

$$\begin{aligned}\Phi_{\mathcal{E}} &= \oiint_{\text{can}} \vec{\mathcal{E}} \cdot d\vec{a} \\ &= \iint_{\text{top}} \vec{\mathcal{E}} \cdot d\vec{a} + \iint_{\text{side}} \vec{\mathcal{E}} \cdot d\vec{a} + \iint_{\text{bot}} \vec{\mathcal{E}} \cdot d\vec{a} \\ &= \mathcal{E} \iint_{\text{top}} da + 0 + \mathcal{E} \iint_{\text{bot}} da \\ &= 2\mathcal{E}a \end{aligned} \tag{3.20}$$

The total charge enclosed

$$Q = \sigma a. \tag{3.21}$$

Substitute flux and charge into the Eq. 3.12 Gauss's law to get

$$\epsilon_0 2\mathcal{E}a = \sigma a \tag{3.22}$$

and conclude

$$\epsilon_0 \mathcal{E} = \frac{\sigma}{2}, \tag{3.23}$$

which agrees with the Eq. 2.69 electric field obtained by direct integration of Coulomb's law. The electrical field is independent of distance because an infinite, homogeneous charge distribution looks the same to a test charge at any distance! ■

3.2.3 Example: Spherical Shell Charge

Consider a spherical shell of radius R with charge density σ, as in Fig. 3.5.

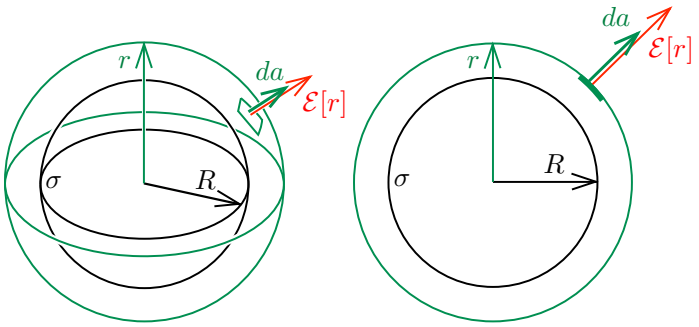


Figure 3.5: Convenient Gaussian surface for a spherical shell of charge is a concentric shell, shown in perspective and cross section.

By symmetry, the electric field $\vec{\mathcal{E}}[\vec{r}] = \mathcal{E}[r]\hat{r}$. While Gauss's law works for any surface, a concentric spherical shell mirrors the symmetry of the problem.

The electric flux through the surface

$$\begin{aligned}\Phi_{\mathcal{E}} &= \oiint_a \vec{\mathcal{E}} \cdot d\vec{a} \\ &= \mathcal{E}[r] \oiint_a da \\ &= \mathcal{E}[r] 4\pi r^2 \end{aligned} \tag{3.24}$$

If the Gaussian surface is outside the shell, $r > R$, and the total charge enclosed

$$Q = \sigma 4\pi R^2. \tag{3.25}$$

Substitute flux and charge into the Eq. 3.12 Gauss's law to get

$$\epsilon_0 \mathcal{E}[r] 4\pi r^2 = \sigma 4\pi R^2 \tag{3.26}$$

and conclude

$$\epsilon_0 \mathcal{E}[r] = \sigma \frac{R^2}{r^2} = \frac{\sigma 4\pi R^2}{4\pi r^2} = \frac{Q}{4\pi r^2}, \tag{3.27}$$

which is the electric field if all the charge on the shell were concentrated at its center!

Alternately, if the Gaussian surface is inside the shell, $r < R$, and the total charge enclosed

$$Q = 0. \tag{3.28}$$

Equate flux and charge

$$\mathcal{E}[r] 4\pi r^2 = 0 \tag{3.29}$$

to conclude

$$\mathcal{E}[r] = 0, \tag{3.30}$$

so the electric field inside the shell vanishes! The vanishing of electric fields inside spherical shells is a very sensitive test of the inverse square nature of Coulomb's law. ■

3.2.4 Example: Spherical Ball Charge

Consider a solid ball of charge, as in Fig. 3.6. Assume a volume charge density ρ with SI unit of

$$\text{unit}[\rho] = \frac{\text{C}}{\text{m}^3}. \tag{3.31}$$

Like the spherical shell charge of Example 3.2.3, concentric spherical shells are convenient Gaussian surfaces for the ball charge, and the exterior case $r > R$ is the same, so that the exterior field is that of a point charge at the center. For the interior case $r < R$, the flux is still

$$\Phi_{\mathcal{E}} = \oiint_a \vec{\mathcal{E}} \cdot d\vec{a} = \mathcal{E}[r] \oiint_a da = \mathcal{E}[r] 4\pi r^2 \tag{3.32}$$

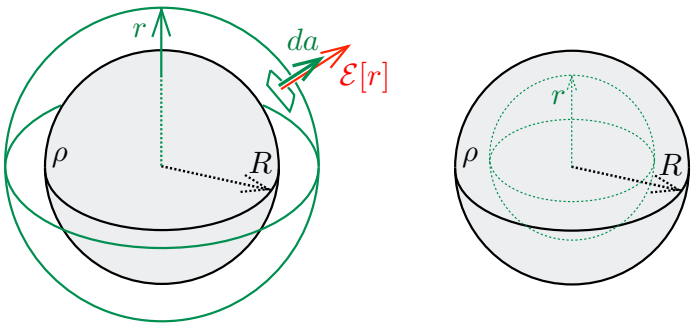


Figure 3.6: Convenient Gaussian surfaces for the interior and exterior of a solid ball of charge are concentric shells.

but the total charge enclosed *by the Gaussian surface* is

$$Q = \rho \frac{4}{3}\pi r^3. \tag{3.33}$$

Substitute flux and charge into the Eq. 3.12 Gauss's law to get

$$\epsilon_0 \mathcal{E}[r] 4\pi r^2 = \rho \frac{4}{3}\pi r^3 \tag{3.34}$$

and conclude

$$\epsilon_0 \mathcal{E}[r] = \frac{1}{3}\rho r, \tag{3.35}$$

so the radial electric field vanishes at the ball's center, which symmetry demands (else which way would it point?), and increases proportionally until the surface of the ball, where it continuously decreases as the inverse square of the distance from the center. ■

3.3 Newton's Shell Theorem

The superb theorems of Example 3.2.3 were first proven by Isaac Newton in the context of gravity. Following Newton, consider a point inside a spherical shell of charge, as in Fig. 3.7.

Nearer but smaller surface charge balances farther but larger surface charge, so the electric field cancels in any direction. Quantitatively, the ratio of the electric field magnitudes due to diametrically opposed surface areas da and dA is

$$\frac{dq/r^2}{dQ/R^2} = \frac{dq}{dQ}\frac{R^2}{r^2} = \frac{da}{dA}\frac{R^2}{r^2} = \frac{da'/\cos\alpha}{dA'/\cos\alpha}\frac{R^2}{r^2} = \frac{da'}{dA'}\frac{R^2}{r^2} = 1, \tag{3.36}$$

in the limit in which the areas shrink to zero. For Proposition 70 of his *Principia*, Newton wrote [1], "If to every point of a spherical surface there tend equal

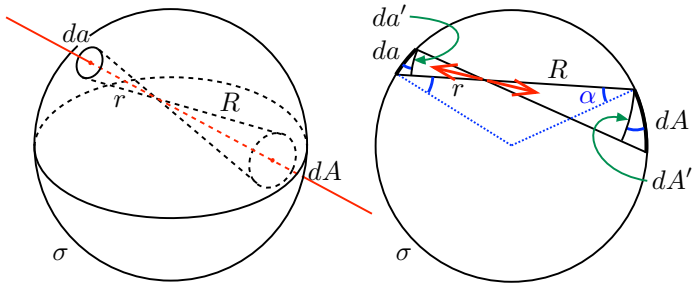


Figure 3.7: Electric field inside a spherical shell of charge cancels in pairs in any direction, as shown in perspective and cross section.

centripetal forces decreasing as the square of the distances from those points, I say, that a corpuscle placed within that surface will not be attracted by those forces any way."

3.4 Problems

1. For each of the electric fields below, find the electric flux through the area $\vec{a} = d^2(\hat{x} - 2\hat{y} + 3\hat{z})$ using the Appendix B dot product formulas.
   (a) $\vec{\mathcal{E}} = \mathcal{E}_0\hat{x}$.
   (b) $\vec{\mathcal{E}} = 3\mathcal{E}_0\hat{y}$.
   (c) $\vec{\mathcal{E}} = 3\mathcal{E}_0\hat{x} + 2\mathcal{E}_0\hat{z}$.
2. Find the electric flux through *each* face of a cube containing a charge Q in the following places.
   (a) The center.
   (b) One corner. (Hint: Stack 8 identical cubes together with the lone charge at the center of the stack.)
3. What is the electric flux through a butterfly net if its radius R circular frame is perpendicular to a constant electric field $\vec{\mathcal{E}}_0$?
4. Use Gauss's law for gravity, Eq. 3.15 with Eq. 3.14, to prove that the gravitational field of any *spherical* mass is the same as that of a point mass at its center.
5. Consider a thick spherical shell of inner radius a, outer radius b, and volume charge density ρ.
   (a) Use Gauss's law to find the electric field $\vec{\mathcal{E}}$ everywhere, in all three regions.
   (b) Qualitatively graph the electric field magnitude $\mathcal{E}$ as a function of radial distance r from the shell's center.
6. Consider a thick, infinitely long, cylinder or "pipe" of inner radius a, outer radius b, and volume charge density ρ.
   (a) Use Gauss's law to find the electric field $\vec{\mathcal{E}}$ everywhere, in all three regions.
   (b) Qualitatively graph the electric field magnitude $\mathcal{E}$ as a function of the perpendicular separation s from the cylinder's axis.

Chapter 4

Biot-Savart

Biot-Savart's law relates electric currents to magnetic fields and the deflecting aspect of electromagnetism.

4.1 Magnetic Force

Coulomb's law gives the force between two charges at rest. However, the forces between two charges is complicated and enriched by motion, as in Fig. 4.1. Such force pairs need not point in opposite directions nor lie along the line joining the charges.

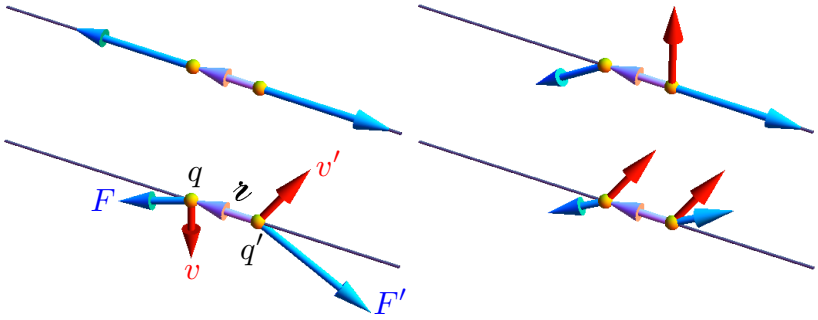


Figure 4.1: Forces $\vec{F}$ and $\vec{F}'$ on charges q and q' moving with constant velocities $\vec{v}$ and $\vec{v}\,'$ need not be opposite nor along the line joining the charges.

Consider the force on a test charge q moving with constant velocity $\vec{v}$ due to a source charge q' moving with constant velocity $\vec{v}\,'$. There is still an "electric" component $\vec{F}_{\mathcal{E}}$ to the force on the test charge that does not depend on its velocity,

$$\epsilon_0 \vec{F}_{\mathcal{E}} \approx q \frac{q'}{4\pi \boldsymbol{r}^2} \hat{\boldsymbol{r}}, \quad v' \ll c, \tag{4.1}$$

where $\vec{\boldsymbol{\imath}}$ is the displacement of the test charge q relative to the source charge q', as usual. However, there is also a "magnetic" component $\vec{F}_{\mathcal{B}}$ to the force on the test charge that *does* depend on its velocity,

$$\vec{F}_{\mathcal{B}} \approx \frac{\vec{v}}{c} \times \left(\frac{\vec{v}\,'}{c} \times \vec{F}_{\mathcal{E}} \right) \approx \mu_0 \vec{v} \times \left(\vec{v}\,' \times q \frac{q'}{4\pi \boldsymbol{\imath}^2} \right), \quad v' \ll c, \tag{4.2}$$

as light speed $c = 1/\sqrt{\epsilon_0 \mu_0}$. These expressions are exact when the speeds are zero and are good approximations when the speeds are small compared to light speed. (They neglect a relativistic compression of the field lines in the direction of motion.)

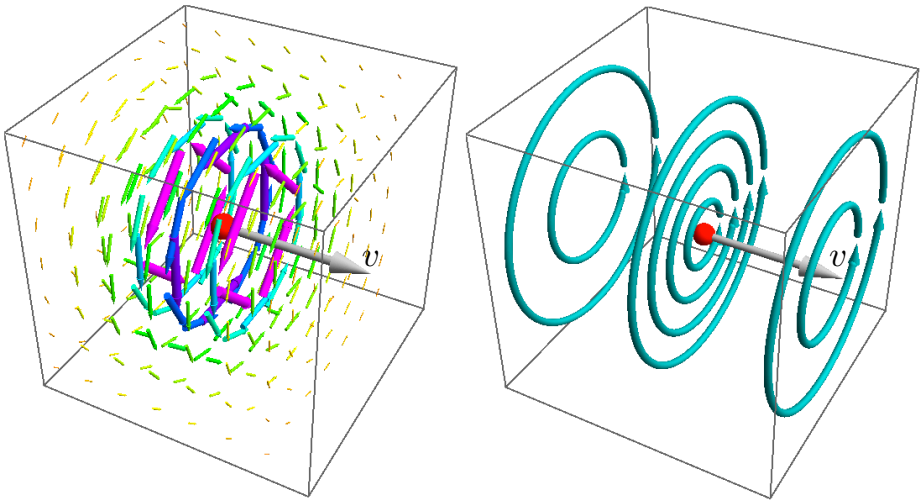


Figure 4.2: Arrows and lines represent the magnetic field $\vec{\mathcal{B}}$ circulating around a positive charge moving with constant velocity $\vec{v}$.

4.2 Magnetic Field

As with the electric force law, isolate the source quantities and decompose the Eq. 4.3 magnetic force law into two parts by introducing the **magnetic field** $\vec{\mathcal{B}}$ via

$$\vec{F}_{\mathcal{B}} = q\vec{v} \times \vec{\mathcal{B}}, \tag{4.3}$$

where

$$\vec{\mathcal{B}} \approx \mu_0 \epsilon_0 \vec{v}\,' \times \vec{\mathcal{E}} \approx \mu_0 \vec{v}\,' \times \frac{q'}{4\pi \boldsymbol{\imath}^2} \hat{\boldsymbol{\imath}}, \quad v' \ll c. \tag{4.4}$$

The cross product in the Eq. 4.3 force means that magnetism *deflects* charges perpendicular to the field and perpendicular to the motion. Since the force is

perpendicular to every infinitesimal displacement, the magnetic force does no work on charges. The cross product in the Eq. 4.4 field implies that the magnetic field of moving charge *circulates* around its velocity vector, as in Fig. 4.2. If the thumb of a right hand points in a charge's direction of motion, then the fingers curl like its magnetic field lines.

The magnetic field has SI unit of **Tesla**,

$$\text{unit}[\mathcal{B}] = \frac{\text{N}}{\text{C}}\frac{\text{s}}{\text{m}} = \text{T}. \tag{4.5}$$

Earth's surface magnetic field (due to electrical currents in its liquid outer core) ranges from 30μT to 60μT, while a typical refrigerator magnet's field (due to atomic currents of unpaired electron spins) is about 5mT. The letter "$\mathcal{B}$" representing the magnetic field is a kind of mirror image of the letter "$\mathcal{E}$" representing the electric field; side-by-side they fuse into a logo "$\mathcal{EB}$" for electromagnetism.

The *total* force on a test charge q in electric $\vec{\mathcal{E}}$ and magnetic $\vec{\mathcal{B}}$ fields is the superposition

$$\vec{F} = \vec{F}_{\mathcal{E}} + \vec{F}_{\mathcal{B}} = q\vec{\mathcal{E}} + q\vec{v} \times \vec{\mathcal{B}} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}), \tag{4.6}$$

which is the **Lorentz force law**, one of the cornerstones of electromagnetism. It describes a wide variety of interesting motions, such as those in Fig. 4.3 and Fig. 4.4.

4.2.1 Example: Cyclotron Motion

Inject a particle of mass m and positive charge $q > 0$ into a constant magnetic field $\vec{\mathcal{B}}$. Combine the Eq. 4.3 force law with Newton's law of motion to get

$$m\vec{a} = \vec{F} = q\vec{v} \times \vec{\mathcal{B}}. \tag{4.7}$$

A magnetic field perpendicular to the initial velocity deflects the particle in uniform circular motion of radius R,

$$-m\frac{v^2}{R} \approx F_r = -qv\mathcal{B}, \quad v \ll c. \tag{4.8}$$

The corresponding **cyclotron frequency** is

$$\omega_c = \frac{v}{R} \approx \frac{q\mathcal{B}}{m}, \quad v \ll c. \tag{4.9}$$

An additional magnetic field component parallel to the velocity deflects the particle in a helix. In this way, electrons and protons from the **solar wind** spiral around Earth's dipole magnetic field and are thereby focussed toward the polar regions where they collide and excite oxygen and nitrogen atoms, which then de-excite to produce beautiful **auroral** light shows. ■

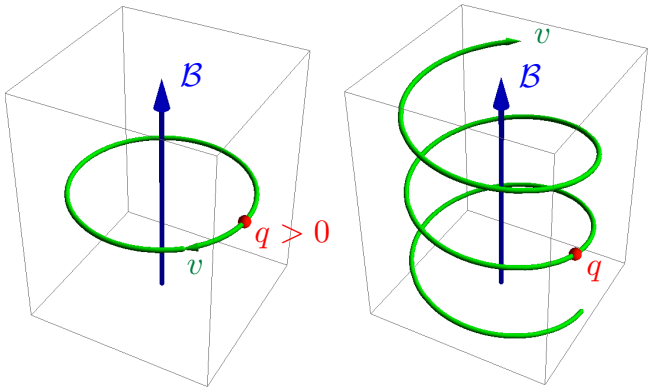


Figure 4.3: Positive charges circulate around magnetic fields in circles or helices depending on the their initial velocities.

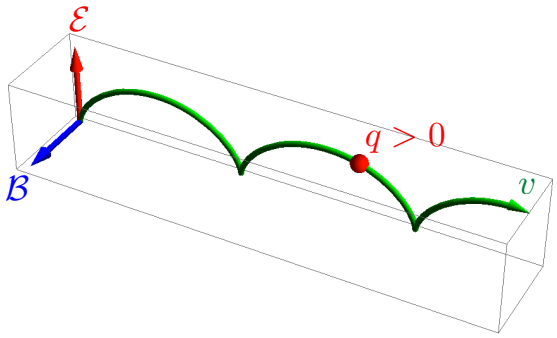


Figure 4.4: Positive charge executes a cycloidal orbit in crossed electric and magnetic fields.

4.2.2 Example: Wire Deflection

Consider a wire segment of length ℓ carrying a current I in a constant magnetic field $\vec{\mathcal{B}}$. The magnetic deflection force on an infinitesimal length $d\vec{\ell}$ is

$$d\vec{F}_{\mathcal{B}} = dQ\vec{v} \times \vec{\mathcal{B}} = dQ\frac{d\vec{\ell}}{dt} \times \vec{\mathcal{B}} = \frac{dQ}{dt}d\vec{\ell} \times \vec{\mathcal{B}} = Id\vec{\ell} \times \vec{\mathcal{B}}, \tag{4.10}$$

and so the force on the entire wire segment is

$$\vec{F}_{\mathcal{B}} = \int d\vec{F}_{\mathcal{B}} = \int_{\ell} Id\vec{\ell} \times \vec{\mathcal{B}} = I\left(\int_{\ell} d\vec{\ell}\right) \times \vec{\mathcal{B}} = I\vec{\ell} \times \vec{\mathcal{B}}, \tag{4.11}$$

where $\vec{\ell}$ points in the direction of the current. ■

4.3 Biot-Savart's Law

In practice, the sources of magnetic fields are often electrical currents, as in Fig. 4.5. If in time dt charge $dq' = I dt$ move distances $d\vec{\ell} = \vec{v}'dt$, then

$$\frac{d\vec{\ell}}{dq'} = \frac{\vec{v}'}{I} \tag{4.12}$$

or

$$\vec{v}'dq' = Id\vec{\ell}. \tag{4.13}$$

Thus, the corresponding magnetic field

$$d\vec{\mathcal{B}} \approx \mu_0 \vec{v}' \times \frac{dq'}{4\pi \boldsymbol{\imath}^2}\hat{\boldsymbol{\imath}}, \quad v' \ll c, \tag{4.14}$$

or

$$d\vec{\mathcal{B}} = \mu_0 \frac{Id\vec{\ell} \times \hat{\boldsymbol{\imath}}}{4\pi \boldsymbol{\imath}^2}, \tag{4.15}$$

and the total magnetic field due to the entire current

$$\vec{\mathcal{B}} = \int d\vec{\mathcal{B}} = \mu_0 \int_\ell \frac{Id\vec{\ell} \times \hat{\boldsymbol{\imath}}}{4\pi \boldsymbol{\imath}^2}, \tag{4.16}$$

which is **Biot-Savart's law**. While the Eq. 4.4 magnetic field of a moving point charge is approximate, Biot-Savart's law for the magnetic field due to a current is exact.

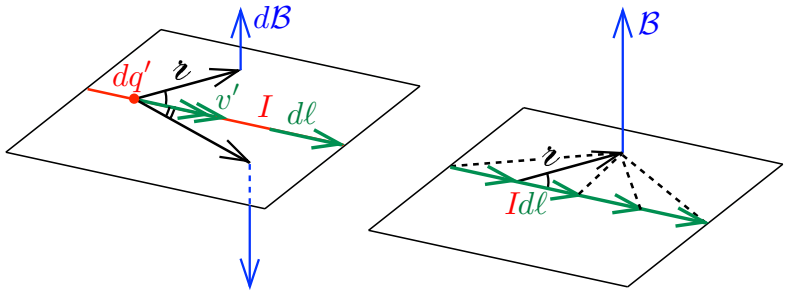


Figure 4.5: Magnetic fields $d\vec{\mathcal{B}}$ due to charges $dq' = Idt$ moving distances $d\vec{\ell} = \vec{v}'dt$ sum to a total field $\vec{\mathcal{B}}$.

4.3.1 Example: Line Current

Consider the magnetic field of an infinite line current I. By symmetry, assume the current coincides with the z-axis, the field point $\vec{r} = x\hat{x}$ is on the x-axis, and the source point $\vec{r}' = z\hat{z}$ is on the z-axis, as in Fig. 4.6. Let α be the angle between $\vec{r}$ and $\vec{\boldsymbol{\imath}} = \vec{r} - \vec{r}'$.

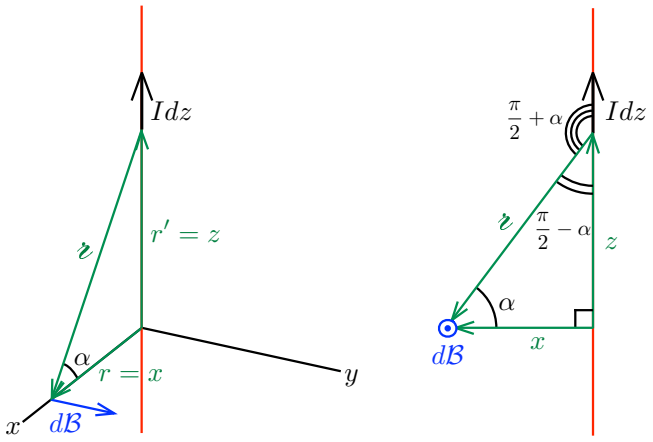


Figure 4.6: A long straight current and the geometry of the position triangle rotated into the plane of the page.

By symmetry, the Eq. 4.16 Biot-Savart's law implies

$$\mathcal{B} = \mathcal{B}_y = \int d\mathcal{B}_y = \mu_0 \int_{-\infty}^{\infty} \frac{I dz \sin[\pi/2 + \alpha]}{4\pi \boldsymbol{\imath}^2}. \tag{4.17}$$

Eliminate the variables z and $\boldsymbol{\imath}$ in favor of the angle α using

$$\boldsymbol{\imath} = \frac{x}{\cos\alpha} = x \sec\alpha \tag{4.18}$$

and

$$z = x \tan\alpha, \tag{4.19}$$

where $z = \pm\infty$ corresponds to $\alpha = \pm\pi/2$. Differentiate to find

$$dz = x(\sec\alpha)^2 \, d\alpha. \tag{4.20}$$

Substitute to get

$$\begin{aligned} \mathcal{B}_y &= \mu_0 \int_{-\pi/2}^{\pi/2} \frac{I \, x(\sec\alpha)^2 d\alpha \, \cos\alpha}{4\pi (x \sec\alpha)^2} \\ &= \mu_0 \frac{I}{4\pi x} \int_{-\pi/2}^{\pi/2} \cos\alpha \, d\alpha \\ &= \mu_0 \frac{I}{2\pi x}. \end{aligned} \tag{4.21}$$

Using the Appendix C cylindrical coordinates $\{s, \phi, z\}$, where s is the perpendicular distance to the z-axis and ϕ is the longitude, generalize this to

$$\vec{\mathcal{B}}[\vec{r}] = \mu_0 \frac{I}{2\pi s} \hat{\phi}. \tag{4.22}$$

The denominator reflects the dilution of the source current over a circle of circumference $2\pi s$. ■

4.3.2 Example: Ring Current

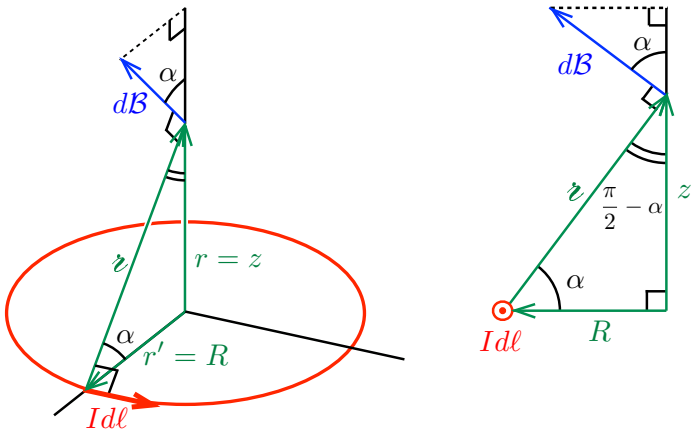


Figure 4.7: A ring current and the geometry of the position triangle rotated into the plane of the page.

Consider the magnetic field $\vec{\mathcal{B}}$ at a distance z on the axis of a circular current I of radius R, as in Fig. 4.7. By symmetry, the horizontal components cancel in pairs, and the Eq. 4.16 Biot-Savart's law implies

$$\mathcal{B}_z = \int d\mathcal{B}_z = \int d\mathcal{B}\cos\alpha = \mu_0 \int_\ell \frac{I d\ell \sin[\pi/2]}{4\pi \ell^2}\cos\alpha. \tag{4.23}$$

Since the angle α and the separation ℓ are constant around the ring,

$$\begin{aligned}\mathcal{B}_z &= \mu_0 \frac{I}{4\pi\ell^2}\left(\int_\ell d\ell\right)\cos\alpha \\ &= \mu_0 \frac{I}{4\pi\ell^2}(2\pi R)\frac{R}{\ell} \\ &= \mu_0 \frac{IR^2}{2(z^2+R^2)^{3/2}},\end{aligned} \tag{4.24}$$

and so

$$\vec{\mathcal{B}}[z] = \mu_0 \frac{IR^2}{2(z^2+R^2)^{3/2}}\hat{z}. \tag{4.25}$$

Far from the ring,

$$\mathcal{B}_z[z \gg R] \approx \mu_0 \frac{IR^2}{2(z^2+0^2)^{3/2}} = \mu_0 \frac{I\pi R^2}{2\pi z^3} = \mu_0 \frac{\mu_{\mathcal{B}}}{2\pi z^3} \propto \frac{1}{z^3}, \tag{4.26}$$

where $\mu_{\mathcal{B}} = I\pi R^2$, and the field decreases as the inverse *cube* of the distance, like that of an electric dipole. ■

4.4 Magnetic Dipole

More generally, a **magnetic dipole** consists of a current I enclosing an area a and characterized by the **magnetic dipole moment**

$$\vec{\mu}_{\mathcal{B}} = I\vec{a}. \tag{4.27}$$

If the fingers of a right hand curl in the direction of the current, then the thumb points in the direction of the magnetic moment. An important idealization is the **point magnetic dipole** characterized by the limit where the area a shrinks to zero as the current I diverges to infinity, such that the product

$$\vec{\mu}_{\mathcal{B}} = \lim_{\substack{I\to\infty \\ a\to 0}} I\vec{a} \tag{4.28}$$

remains finite.

Generalizing the Eq. 4.26 magnetic field on the axis of a ring charge, and in analogy with the Eq. 2.47 dipole electric field, the dipole magnetic field is

$$\vec{\mathcal{B}}[\vec{r}] = \mu_0 \frac{3(\vec{\mu}_{\mathcal{B}}\cdot\hat{r})\hat{r} - \vec{\mu}_{\mathcal{B}}}{4\pi r^3}, \tag{4.29}$$

where $\vec{r} = r\hat{r}$ is an arbitrary point in space. This is a good approximation to the magnetic field of Earth or of a common bar magnet or of a current ring, as in Fig. 4.8. Electrons are electric monopoles, but magnetic dipoles.

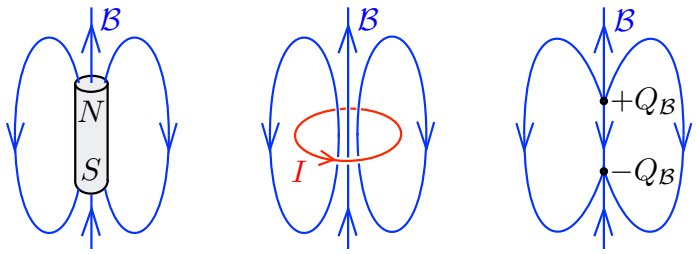


Figure 4.8: A bar magnet and a current ring are magnetic dipoles with common external magnetic fields, shown here schematically.

4.5 Magnetic Monopole

Cut a bar dipole magnet in half and get two dipoles rather than two monopole fields. Do **magnetic monopoles** exist? With the possible exception of the

1982 Valentine Day's Monopole [3], they have never been observed. However, magnetic monoples, also known as **magnetic charges**, are predicted by certain **Grand Unified Theories**. Furthermore, they symmetrize the electromagnetic equations. By analogy with electric charge $Q_{\mathcal{E}}$, describe the electric and magnetic fields of a magnetic charge $Q_{\mathcal{B}}$ moving with constant velocity $\vec{v}'_{\mathcal{B}}$ by

$$\vec{\mathcal{B}} \approx \mu_0 \frac{Q_{\mathcal{B}}}{4\pi \boldsymbol{\ell}^2} \hat{\boldsymbol{\ell}}, \quad v'_{\mathcal{B}} \ll c, \tag{4.30}$$

and

$$\vec{\mathcal{E}} \approx -\vec{v}'_{\mathcal{B}} \times \vec{\mathcal{B}}, \quad v'_{\mathcal{B}} \ll c. \tag{4.31}$$

Describe the force on a magnetic charge $Q_{\mathcal{B}}$ moving with constant velocity $\vec{v}_{\mathcal{B}}$ in external electric and magnetic field by

$$\vec{F} = Q_{\mathcal{B}}(\vec{\mathcal{B}} - \epsilon_0\mu_0 \vec{v}_{\mathcal{B}} \times \vec{\mathcal{E}}). \tag{4.32}$$

The minus signs in these expressions arise from **relativity**. For example, consider an electric charge and a magnetic charge in relative motion, as in Fig. 4.9. Ride with the magnetic charge $Q_{\mathcal{B}}$ and observe a force on the magnetic charge,

$$\vec{F}_{\mathcal{B}\mathcal{B}} = Q_{\mathcal{B}}\vec{\mathcal{B}} \approx \epsilon_0\mu_0 Q_{\mathcal{B}} \vec{v}'_{\mathcal{E}} \times \vec{\mathcal{E}}. \tag{4.33}$$

Ride with the electric charge $Q_{\mathcal{E}}$ and observe the *same* force on the magnetic charge but with a different explanation,

$$\vec{F}_{\mathcal{B}\mathcal{E}} = -\epsilon_0\mu_0 Q_{\mathcal{B}} \vec{v}_{\mathcal{B}} \times \vec{\mathcal{E}} \approx \epsilon_0\mu_0 Q_{\mathcal{B}} \vec{v}'_{\mathcal{E}} \times \vec{\mathcal{E}} \approx \vec{F}_{\mathcal{B}\mathcal{B}}, \tag{4.34}$$

as the relative velocity $\vec{v}_{\mathcal{B}} = -\vec{v}'_{\mathcal{E}}$ (and the relative speed $v_{\mathcal{B}} = v'_{\mathcal{E}}$). Table 4.1 outlines the symmetry between electricity and magnetism.

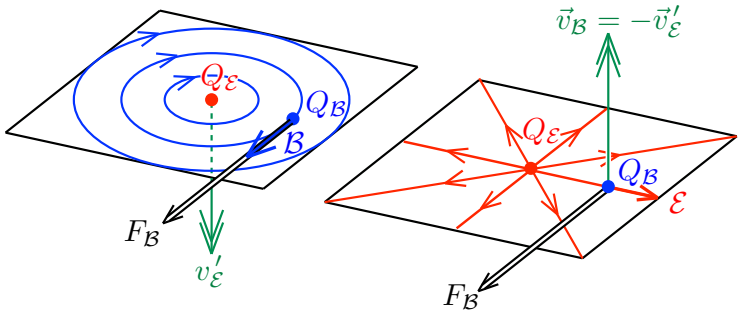


Figure 4.9: Force $\vec{F}_{\mathcal{B}}$ on magnetic charge $Q_{\mathcal{B}}$ due to motion relative to electric charge $Q_{\mathcal{E}}$ from each charge's reference frame.

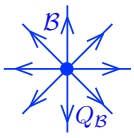
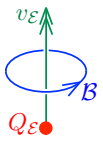
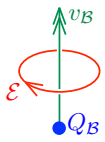
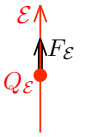
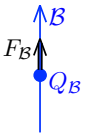
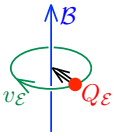
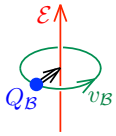
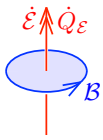
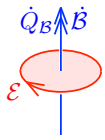
4.6 Magnetic Gauss's Law

In analogy with the Eq. 3.12 electric Gauss's law, the magnetic flux through any closed surface is proportional to the magnetic charge inside,

$$\Phi_{\mathcal{B}} = \mu_0 Q_{\mathcal{B}}. \tag{4.35}$$

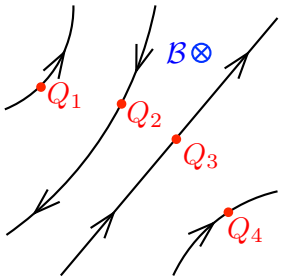
While the electric Gauss's law implies that electric field lines start and stop and electric charges, in the absence of magnetic monopoles, the magnetic Gauss's law implies that magnetic field lines never stop or stop but always circulate, like they do around electric currents. This is the second of the four Maxwell equations that summarize electromagnetism.

Table 4.1: The mutual embrace of electricity and magnetism.

$\epsilon_0 \vec{\mathcal{E}} = \frac{Q_{\mathcal{E}}}{4\pi \boldsymbol{r}^2}\hat{\boldsymbol{r}}$			$\vec{\mathcal{B}} = \mu_0 \frac{Q_{\mathcal{B}}}{4\pi \boldsymbol{r}^2}\hat{\boldsymbol{r}}$
$\vec{\mathcal{B}} \approx +\epsilon_0\mu_0 \vec{v}_{\mathcal{E}} \times \vec{\mathcal{E}}$			$\vec{\mathcal{E}} \approx -\vec{v}_{\mathcal{B}} \times \vec{\mathcal{B}}$
$\vec{F}_{\mathcal{E}} = Q_{\mathcal{E}}\vec{\mathcal{E}}$			$\vec{F}_{\mathcal{B}} = Q_{\mathcal{B}}\vec{\mathcal{B}}$
$\vec{F}_{\mathcal{E}} = +Q_{\mathcal{E}}\vec{v}_{\mathcal{E}} \times \vec{\mathcal{B}}$			$\vec{F}_{\mathcal{B}} = -\epsilon_0\mu_0 Q_{\mathcal{B}} \vec{v}_{\mathcal{B}} \times \vec{\mathcal{E}}$
$\epsilon_0 \Phi_{\mathcal{E}} = Q_{\mathcal{E}}$			$\Phi_{\mathcal{B}} = \mu_0 Q_{\mathcal{B}}$
$\Gamma_{\mathcal{B}} = +\epsilon_0\mu_0\dot{\Phi}_{\mathcal{E}} + \mu_0\dot{Q}_{\mathcal{E}}$			$\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}} - \mu_0\dot{Q}_{\mathcal{B}}$

4.7 Problems

1. If particles move in the directions of the arrows, perpendicular to a constant inward $\otimes$ magnetic field $\vec{\mathcal{B}}$, are their charges Q_n positive, zero, or negative?



2. For each magnetic field, find the force on a charge q moving with velocity $\vec{v} = v_0(\hat{x} + 2\hat{y})$ using the Appendix B cross product formulas.

   (a) $\vec{\mathcal{B}} = \mathcal{B}_0\hat{x}$.

   (b) $\vec{\mathcal{B}} = 2\mathcal{B}_0\hat{y}$.

   (c) $\vec{\mathcal{B}} = 3\mathcal{B}_0\hat{x} + 2\mathcal{B}_0\hat{z}$.

3. A perpendicular distance s separates two parallel currents I and I' flowing in opposite directions.

   (a) Is the force attractive or repulsive?

   (b) What is the force magnitude per unit length between them?

4. A straight line of charge density λ moves parallel to itself at speed v.

   (a) What is the corresponding current I?

   (b) What is the ratio of the corresponding magnetic and electric field magnitudes $\mathcal{B}/\mathcal{E}$?

5. What is the ratio of the magnetic field magnitudes at the center of a circular current I of radius R and at a distance R from a long, straight current I? Neat!

6. What is the magnetic field $\vec{\mathcal{B}}$ at the center of a square loop of side $2R$ carrying a current I ? (Hint: First modify the Sec. 4.3.1 example to find the magnetic field due to one side of the square.)

Chapter 5

Ampère

Part of one of the four Maxwell equations that summarize electromagnetism, Ampère's law provides a new perspective on Biot-Savart's law. It relates magnetic fields on an arbitrary path to the current piercing the path. The relationship is simplest in terms of a quantity called **circulation**.

5.1 Circulation

In hydrodynamics, **volume circulation** Γ_V is the rate per distance (in, say, liters per second per meter) that water flows around a closed loop. If the loop ℓ is everywhere tangent to a flow of velocity $\vec{v}$, then the volume circulation is simply

$$\Gamma_V = v\ell. \tag{5.1}$$

More generally,

$$\Gamma_V = \oint_\ell \vec{v} \cdot d\vec{\ell}. \tag{5.2}$$

A nonzero flux $\Phi_V \neq 0$ indicates the water is diverging (or converging), while a nonzero circulation $\Gamma_V \neq 0$ indicates the water is rotating.

In analogy with hydrodynamical flux, the **magnetic circulation** $\Gamma_\mathcal{B}$ around a closed loop ℓ is

$$\Gamma_\mathcal{B} = \oint_\ell \vec{\mathcal{B}} \cdot d\vec{\ell}. \tag{5.3}$$

The magnetic circulation is the product of the average parallel component of the magnetic field $\langle \mathcal{B}_\parallel \rangle$ and the loop length ℓ,

$$\Gamma_\mathcal{B} = \oint_\ell \vec{\mathcal{B}} \cdot d\vec{\ell} = \oint_\ell \mathcal{B}_\parallel d\ell = \left(\frac{1}{\ell} \oint_\ell \mathcal{B}_\parallel d\ell \right) \ell = \langle \mathcal{B}_\parallel \rangle \ell. \tag{5.4}$$

(Write $\vec{v} \cdot d\vec{\ell} = v_\parallel d\ell$ but $\vec{v} \cdot d\vec{a} = v_\perp da$, because the line vector $d\vec{\ell}$ is parallel to the line element but the area vector $d\vec{a}$ is perpendicular to the area element.)

5.1.1 Example: Line Current Circulation

Consider the magnetic circulation $\mathit{\Gamma}_{\mathcal{B}}$ around a circular loop ℓ of radius s due to a line current I perpendicular to its center. Since both the field $\vec{\mathcal{B}}[\vec{r}] = \mathcal{B}_\phi[s]\hat{\phi}$ and the infinitesimal line elements $d\vec{\ell} = d\ell\,\hat{\phi}$ are longitudinal,

$$\mathit{\Gamma}_{\mathcal{B}} = \oint_\ell \vec{\mathcal{B}}\cdot d\vec{\ell} = \oint_\ell \mathcal{B}_\phi d\ell = \mathcal{B}_\phi[s]\oint_{s=\text{const}} d\ell = \mu_0 \frac{I}{2\pi s} 2\pi s = \mu_0 I \tag{5.5}$$

or

$$\mu_0^{-1}\mathit{\Gamma}_{\mathcal{B}} = I. \tag{5.6}$$

The magnetic circulation is proportional to the current. Indeed, in natural units, where the magnetic constant μ_0 is unity, the magnetic circulation *is* the current. ■

5.2 Ampère's Law

Generalize the Eq. 5.6 proportionality between circulation and current to an arbitrary closed loop, which need not be planar, by decomposing it into infinitesimal segments perpendicular, equidistant, and parallel to the current, as in Fig. 5.1. Using the Appendix C cylindrical coordinates $\{s, \phi, z\}$, the circulation around the arbitrary loop

$$\begin{aligned}
\mathit{\Gamma}_{\mathcal{B}} &= \oint_\ell \vec{\mathcal{B}}\cdot d\vec{\ell} \\
&= \int_\perp \vec{\mathcal{B}}\cdot d\vec{\ell} + \int_\odot \vec{\mathcal{B}}\cdot d\vec{\ell} + \int_\parallel \vec{\mathcal{B}}\cdot d\vec{\ell} \\
&= \int_\perp \mathcal{B}_s\,\hat{s}\cdot d\ell\,\hat{\phi} + \int_\odot \mathcal{B}_\phi\,\hat{\phi}\cdot d\ell\,\hat{\phi} + \int_\parallel \mathcal{B}_z\,\hat{z}\cdot d\ell\,\hat{\phi} \\
&= \quad 0 \quad + \int_0^{2\pi} \mu_0 \frac{I}{2\pi s}\, s d\phi + \quad 0 \\
&= \mu_0 \frac{I}{2\pi}\int_0^{2\pi} d\phi \\
\mathit{\Gamma}_{\mathcal{B}} &= \mu_0 I,
\end{aligned} \tag{5.7}$$

where the perpendicular and parallel components vanish leaving just the equidistant components. Hence, the circulation around the arbitrary loop is the circulation around a circle, and both are proportional to the current current passing through any area bounded by the loop. Generalize this to an arbitrary current distribution by superposition,

$$\mathit{\Gamma}_{\mathcal{B}} = \oint_\ell \vec{\mathcal{B}}\cdot d\vec{\ell} = \oint_\ell \sum_n \vec{\mathcal{B}}_n \cdot d\vec{\ell} = \sum_n \oint_\ell \vec{\mathcal{B}}_n \cdot d\vec{\ell} = \sum_n \mu_0 I_n = \mu_0 I, \tag{5.8}$$

which is **Ampère's law**,

$$\mu_0^{-1}\mathit{\Gamma}_{\mathcal{B}} = I_{\text{thru}}, \tag{5.9}$$

where the additional subscript "thru" is a reminder that I now represents the total, algebraic sum of the currents passing through any area a bounded by the loop ℓ. If the fingers of a right hand curl in the direction of the circulation, the thumb points in the positive direction of the area.

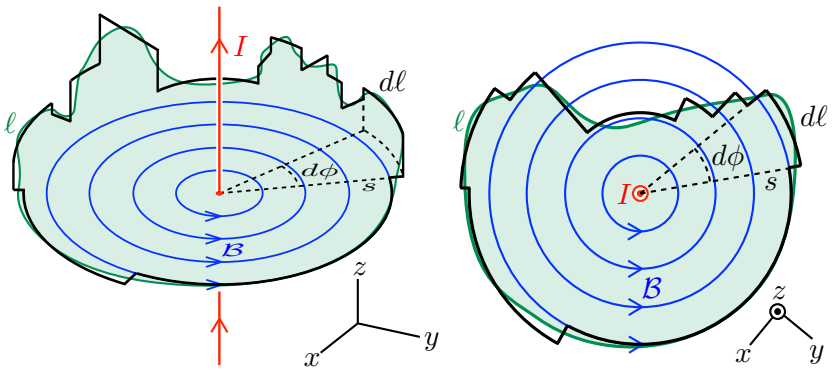


Figure 5.1: Arbitrary loop ℓ decomposed into segments perpendicular, equidistant, and parallel to the current I, shown in perspective and from above.

Ampère's law is to Biot-Savart's law as Gauss's law is to Coulomb's law. Ampère's law is always valid (for constant fields), but it is most useful in cases of high symmetry. Put Ampère's Law to work by using it to find the magnetic fields of symmetric current distributions.

5.2.1 Example: Current Cylinder

Consider the magnetic field $\vec{\mathcal{B}}$ a distance s from the axis of the Fig. 5.2 infinite current cylinder of radius R, an idealization of a current-carrying copper wire. Assume an area current density

$$J_x = \frac{dI}{da_z} = \frac{d^2Q}{dx\,dy\,dt} = \frac{d^2Q}{dx\,dy\,dz}\frac{dz}{dt} = \rho v_z \tag{5.10}$$

with SI unit of

$$\text{unit}[J] = \frac{\text{A}}{\text{m}^2}. \tag{5.11}$$

By symmetry, the magnetic field $\vec{\mathcal{B}}[\vec{r}] = \mathcal{B}[s]\hat{\phi}$. While Ampère's law works for any loop, a circular loop of radius s concentric with the cylinder mirrors the symmetry of the problem. The magnetic circulation around the loop

$$\Gamma_{\mathcal{B}} = \oint_{\text{circle}} \vec{\mathcal{B}} \cdot d\vec{\ell} = \mathcal{B}[s] \oint_{s=\text{const}} d\ell = \mathcal{B}\, 2\pi s.$$

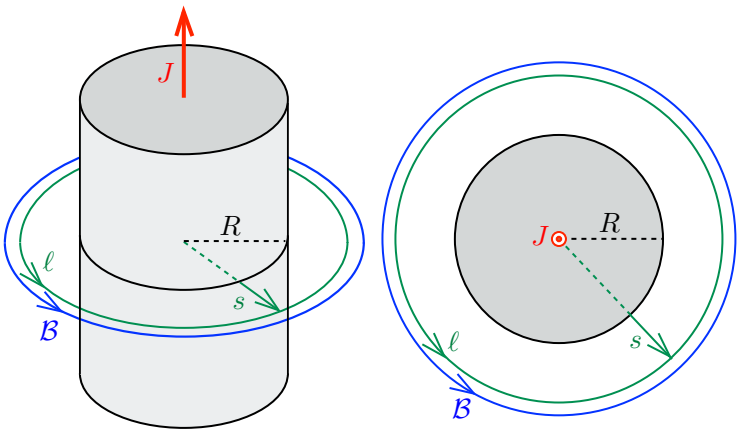


Figure 5.2: Convenient Ampèrean loop for an infinite plane current is rectangular because field is parallel to the plane and perpendicular to the current.

If the Ampèrean loop is outside the cylinder, $s > R$, total current through the loop

$$I = J\pi R^2. \tag{5.12}$$

Substitute circulation and current into the Eq. 5.9 Ampère's law to get

$$\mathcal{B}\, 2\pi s = \mu_0 J\pi R^2 \tag{5.13}$$

and conclude

$$\mathcal{B} = \mu_0 J\frac{R^2}{2s} = \mu_0 \frac{J\pi R^2}{2\pi s} = \mu_0 \frac{I}{2\pi s}, \tag{5.14}$$

which is the magnetic field if all the current through the cylinder were concentrated along it axis.

Alternately, if the Ampèrean loop is inside the cylinder, $s < R$, total current through the loop

$$I = J\pi s^2. \tag{5.15}$$

Equate circulation and current

$$\mathcal{B}\, 2\pi s = \mu_0 J\pi s^2 \tag{5.16}$$

and conclude

$$\mathcal{B} = \mu_0 \frac{1}{2} Js, \tag{5.17}$$

so the magnetic field increases proportionally with separation s from the axis. ■

5.2.2 Example: Current Plane

Consider the magnetic field $\vec{\mathcal{B}}$ a distance z above the Fig. 5.3 infinite plane current flowing in the x-direction. Assume a linear current density

$$K_x = \frac{dI}{dy} = \frac{d^2Q}{dy\,dt} = \frac{d^2Q}{dy\,dx}\frac{dx}{dt} = \sigma v_x \tag{5.18}$$

with SI unit of

$$\text{unit}[K] = \frac{\text{A}}{\text{m}}. \tag{5.19}$$

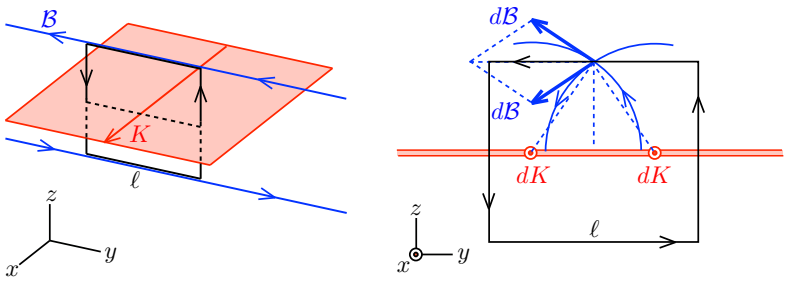


Figure 5.3: Convenient Ampèrean loop for an infinite plane current is rectangular because field is parallel to the plane and perpendicular to the current.

By symmetry, the magnetic field $\vec{\mathcal{B}}[\vec{r}] = \mp\mathcal{B}[z]\hat{y}$ for $z \gtrless 0$, so that the field is parallel to the plane and perpendicular to the current. While Ampère's law works for any loop, a finite rectangular loop of length ℓ straddling the plane mirrors the symmetry of the problem. The magnetic circulation around the loop

$$\begin{aligned}
\Gamma_{\mathcal{B}} &= \oint_{\text{rectangle}} \vec{\mathcal{B}} \cdot d\vec{\ell} \\
&= \int_{\text{top}} \vec{\mathcal{B}} \cdot d\vec{\ell} + \int_{\text{left}} \vec{\mathcal{B}} \cdot d\vec{\ell} + \int_{\text{bot}} \vec{\mathcal{B}} \cdot d\vec{\ell} + \int_{\text{right}} \vec{\mathcal{B}} \cdot d\vec{\ell} \\
&= +\mathcal{B}\ell + 0 + \mathcal{B}\ell + 0 \\
&= 2\mathcal{B}\ell.
\end{aligned} \tag{5.20}$$

The total current through the loop

$$I = K\ell. \tag{5.21}$$

Substitute circulation and current into the Eq. 5.9 Ampère's law to get

$$2\mathcal{B}\ell = \mu_0 K\ell \tag{5.22}$$

and conclude

$$\mathcal{B} = \mu_0 \frac{K}{2}. \tag{5.23}$$

which is exact for an infinite plane current and a good approximation near the middle of a large but finite plane current. The 2 in the denominator reflects the two sides of the plane. ■

Table 5.1 summarizes common current density symbols, and Table 5.2 summarizes the magnetic field of line and plane currents. The fields closely follow the geometry.

Table 5.1: One and two-dimensional current densities.

Dimension	Symbol	SI Unit
line	$\vec{K} = d\vec{I}/d\ell_\perp = \sigma\vec{v}$	$\mathrm{A/m}^1$
area	$\vec{J} = d\vec{I}/da_\perp = \rho\vec{v}$	$\mathrm{A/m}^2$

Table 5.2: Geometry determines the fields of simple current distributions.

Current Distribution	Magnetic Field
line	$\mathcal{B}_\phi = \mu_0 \dfrac{I}{2\pi s}$
area	$\mathcal{B}_\parallel = \mu_0 \dfrac{K}{2}$

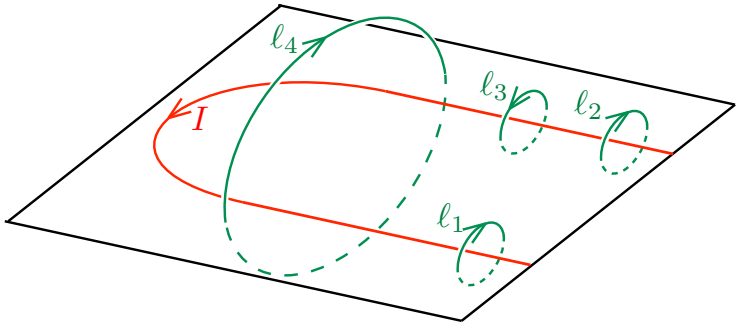
5.3 Problems

1. A distance s separates two parallel currents I. Find the magnetic fields $\vec{\mathcal{B}}$ at the following places.

   (a) A distance s directly above the left current.

   (b) A distance $s/2$ directly above the midline between the currents.

2. What are the magnetic circulations $\Gamma_{\mathcal{B}}$ around the closed loops ℓ_n due to the electrical current I?



3. What is the circulation Γ_v per unit area of the linear velocity $\vec{v}$ of a bicycle wheel rim of radius R spinning with angular velocity ω? (Hint: The result should be a radius-independent quantity called "curl".)

4. A thick, long, tubular wire has inner radius a, outer radius b, and current density $\vec{J}$ in the direction of the wire's axis.

   (a) Use Ampère's law to find the magnetic field $\vec{\mathcal{B}}$ everywhere, in all three regions.

   (b) Qualitatively graph the magnetic field magnitude $\mathcal{B}$ as a function of the perpendicular separation s from the cylinder's axis.

5. A coaxial cable consists of a copper wire of radius R surrounded by a tubular conducting shield of inner radius a and outer radius b. The wire and the shield carry equal but opposite currents $\pm I$ uniformly distributed over their cross sections.

   (a) Use Ampère's law to find the magnetic field $\vec{\mathcal{B}}$ everywhere, in all four regions.

   (b) Qualitatively graph the magnetic field magnitude $\mathcal{B}$ as a function of the perpendicular separation s from the cylinder's axis.

Chapter 6

Faraday

Faraday's law relates changing magnetic fields to electric fields. Indeed, a changing magnetic flux induces a circulating electric field.

6.1 Magnetic Flux and Electric Circulation

Consider a magnetic charge incident on a wire loop. As it approaches the loop, more and more of its magnetic flux pierces the loop. Meanwhile, the magnetic charge's circulating electric field grows stronger and stronger at the loop. Calculate the rate of change with time of its magnetic flux through the loop and compare to its circulating electric field at the loop.

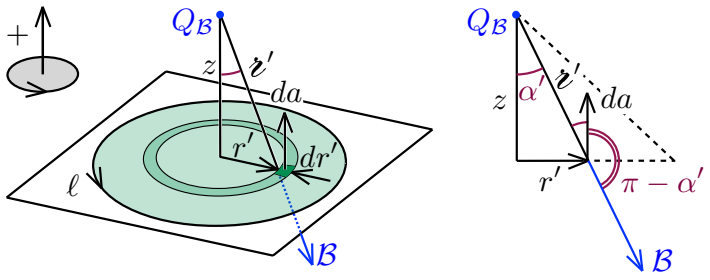


Figure 6.1: Geometry for magnetic charge flux calculation.

To calculate the flux, assume the magnetic charge $Q_{\mathcal{B}} > 0$ at a height $z > 0$ on the axis of a circular wire loop of radius r, as in Fig. 6.1. If up is positive so that the sum of the area elements $d\vec{a} = da\,\hat{z}$ at constant radii $r' < r$ are infinitesimal concentric rings of area $\int da = 2\pi r' dr'$, then the magnetic flux

through the loop is

$$\begin{aligned}\Phi_{\mathcal{B}} &= \iint_a \vec{\mathcal{B}} \cdot d\vec{a} \\ &= \iint_a \mathcal{B}\, da \cos[\pi - \alpha'] \\ &= \int_0^r \mu_0 \frac{Q_{\mathcal{B}}}{4\pi \mathscr{r}'^2} (2\pi r' dr')(-\cos\alpha') \\ &= -\mu_0 \frac{Q_{\mathcal{B}}}{2} \int_0^r \frac{\cos\alpha'\, r' dr'}{\mathscr{r}'^2}. \end{aligned} \tag{6.1}$$

Eliminate the variables r' and $\mathscr{r}'$ in favor of the angle α' using

$$\mathscr{r}' = z \sec\alpha' \tag{6.2}$$

and

$$r' = z \tan\alpha', \tag{6.3}$$

where α' ranges from 0 to α as r' ranges from 0 to r. Differentiate to find

$$dr' = z(\sec\alpha')^2\, d\alpha'. \tag{6.4}$$

Substitute to get

$$\begin{aligned}\Phi_{\mathcal{B}} &= -\mu_0 \frac{Q_{\mathcal{B}}}{2} \int_0^\alpha \frac{\cos\alpha'(z\tan\alpha')z(\sec\alpha')^2 d\alpha'}{(z\sec\alpha')^2} \\ &= -\mu_0 \frac{Q_{\mathcal{B}}}{2} \int_0^\alpha \sin\alpha'\, d\alpha' \\ &= -\mu_0 \frac{Q_{\mathcal{B}}}{2} (-\cos\alpha')\Big|_0^\alpha \\ &= -\mu_0 \frac{Q_{\mathcal{B}}}{2} (1 - \cos\alpha) < 0. \end{aligned} \tag{6.5}$$

The Fig. 6.2 geometry implies

$$\cos\alpha = \frac{z}{\mathscr{r}} = \frac{z}{\sqrt{z^2 + r^2}}, \tag{6.6}$$

and so the magnetic flux

$$\begin{aligned}\Phi_{\mathcal{B}}[z > 0] &= -\mu_0 \frac{Q_{\mathcal{B}}}{2} \left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right) < 0 \\ &= -\mu_0 \frac{Q_{\mathcal{B}}}{2} \left(1 - z(z^2 + r^2)^{-1/2}\right). \end{aligned} \tag{6.7}$$

If the magnetic charge moves downward at a speed $\dot{z} = -v < 0$, where the over-dot is Newton's notation for time derivative, the magnetic flux through the

loop changes with time at the rate

$$\begin{aligned}\dot{\Phi}_{\mathcal{B}} = \frac{d\Phi_{\mathcal{B}}}{dt} &= -\mu_0 \frac{Q_{\mathcal{B}}}{2}\left(0 - \dot{z}(z^2+r^2)^{-1/2} + z\frac{1}{2}(z^2+r^2)^{-3/2}(2z\dot{z}+0)\right) \\ &= -\mu_0 \frac{Q_{\mathcal{B}}}{2}\left(-\dot{z}(z^2+r^2-z^2)(z^2+r^2)^{-3/2}\right) \\ &= -\mu_0 \frac{Q_{\mathcal{B}}}{2}\frac{vr^2}{(z^2+r^2)^{3/2}} < 0. \end{aligned} \tag{6.8}$$

The flux is negative and becoming more negative, decreasing in value but increasing in magnitude, as the monopole approaches the wire loop from above.

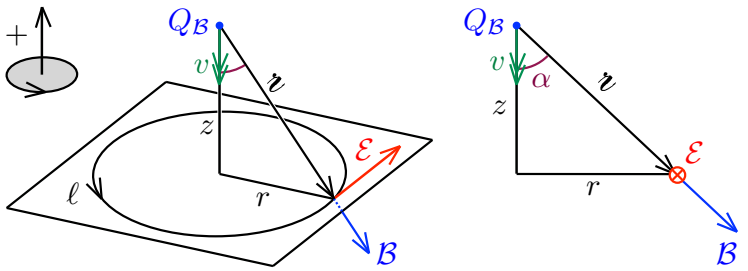


Figure 6.2: Geometry for magnetic charge circulation calculation.

Meanwhile, the moving charge's Eq. 4.31 electric field

$$\vec{\mathcal{E}} \approx -\vec{v}\times\vec{\mathcal{B}} \approx -\vec{v}\times\mu_0\frac{Q_{\mathcal{B}}}{4\pi\boldsymbol{\imath}^2}\hat{\boldsymbol{\imath}} \tag{6.9}$$

produces in the loop the the electric circulation

$$\begin{aligned}\Gamma_{\mathcal{E}} &= \oint_\ell \vec{\mathcal{E}}\cdot d\vec{\ell} \\ &= \mathcal{E}\oint_\ell d\ell \\ &\approx v\left(\mu_0\frac{Q_{\mathcal{B}}}{4\pi\boldsymbol{\imath}^2}\right)\sin\alpha\oint_\ell d\ell \\ &= v\left(\mu_0\frac{Q_{\mathcal{B}}}{4\pi\boldsymbol{\imath}^2}\right)\left(\frac{r}{\boldsymbol{\imath}}\right)(2\pi r) \\ &= \mu_0\frac{Q_{\mathcal{B}}}{2}\frac{vr^2}{\boldsymbol{\imath}^3} \\ &= \mu_0\frac{Q_{\mathcal{B}}}{2}\frac{vr^2}{(z^2+r^2)^{3/2}} > 0, \end{aligned} \tag{6.10}$$

and so the circulation increases as the charge approaches. Indeed, by Eq. 6.8, the circulation is proportional to the rate of change of the magnetic flux,

$$\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}. \tag{6.11}$$

6.2 Faraday's Law

More generally, the circulation of the electric field around *any* closed loop ℓ is minus the rate of change with time of the magnetic flux through *any* area a bounded by the loop,

$$\oint_l \vec{\mathcal{E}} \cdot d\vec{\ell} = \Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}} = -\frac{d\Phi_{\mathcal{B}}}{dt} = -\frac{d}{dt}\iint_a \vec{\mathcal{B}} \cdot d\vec{a}, \tag{6.12}$$

which is **Faraday's law**.

Supposedly, in the 1800s, Michael Faraday was often asked, "What good is electromagnetism?" Sometimes he replied, "What good is a new born baby?" Other times he replied,"Someday you'll tax it." Faraday's experiments laid the foundation for our entire electromagnetic technology.

6.2.1 Example: Electromagnet

Two cylindrical poles of an electromagnetic generate a uniform but decreasing magnetic field magnitude $\vec{\mathcal{B}}[t] = \mathcal{B}_0 - kt$, where the rate $k > 0$, as in Fig. 6.3.

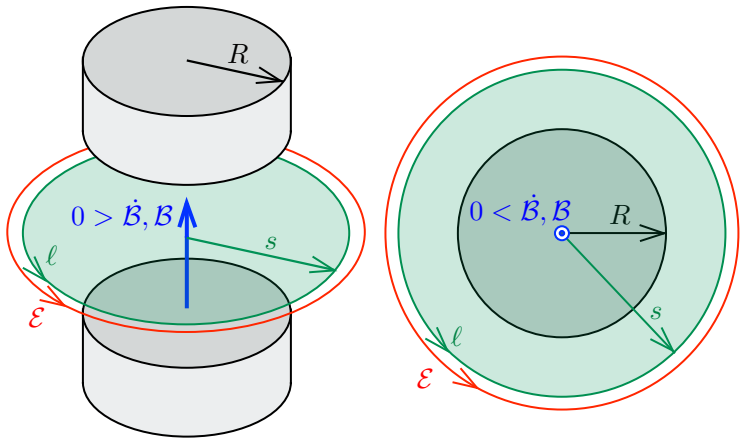


Figure 6.3: A time varying magnetic field $\vec{\mathcal{B}}[t]$ induces a circulating electric field $\vec{\mathcal{E}}[t]$.

By symmetry, the induced electric field $\vec{\mathcal{E}}[\vec{r}] = \mathcal{E}_\phi[s]\hat{\phi}$ circulates around the cylinder. While Faraday's law works for any loop, a circular loop of radius s concentric with the cylinder mirrors the symmetry of the problem. The electric circulation around the loop

$$\Gamma_{\mathcal{E}} = \oint_{\text{circle}} \vec{\mathcal{E}} \cdot d\vec{\ell} = \mathcal{E}[s] \oint_{s=\text{const}} d\ell = \mathcal{E}_\phi \, 2\pi s.$$

If the Faradayen loop is outside the cylinder, $s > R$, the magnetic flux through the loop

$$\Phi_{\mathcal{B}} = \iint_{\text{disk}} \vec{\mathcal{B}} \cdot d\vec{a} = (\pi R^2)(\mathcal{B}_0 - kt), \tag{6.13}$$

and its rate of change with time

$$\dot{\Phi}_{\mathcal{B}} = \frac{d\Phi_{\mathcal{B}}}{dt} = -\pi R^2 k. \tag{6.14}$$

Substitute circulation and rate of change of flux into the Eq. 6.12 Faraday's law to get

$$\mathcal{E}_\phi \, 2\pi s = +\pi R^2 k \tag{6.15}$$

and conclude

$$\mathcal{E}_\phi = \frac{1}{2} k \frac{R^2}{s}, \tag{6.16}$$

so the electric field decreases inversely with separation s from the axis.

Alternately, if the Faradayen loop is inside the cylinder, $s < R$, the magnetic flux through the loop

$$\Phi_{\mathcal{B}} = \iint_{\text{disk}} \vec{\mathcal{B}} \cdot d\vec{a} = (\pi s^2)(\mathcal{B}_0 - kt), \tag{6.17}$$

and its rate of change with time

$$\dot{\Phi}_{\mathcal{B}} = \frac{d\Phi_{\mathcal{B}}}{dt} = -\pi s^2 k. \tag{6.18}$$

Equate circulation and minus rate of change of flux

$$\mathcal{E}_\phi \, 2\pi s = +\pi s^2 k. \tag{6.19}$$

and conclude

$$\mathcal{E}_\phi = \frac{1}{2} k s > 0, \tag{6.20}$$

so the electric field increases proportionally with separation s from the axis, counterclockwise from above in Fig. 6.3. If the Faradayen loops were wires, the induced electric field would drive currents whose upward magnetic fields would oppose the decrease of the magnetic flux through the loops. ■

6.2.2 Example: Electric Generator

A linear **electric generator** consists of a constant magnetic field $\mathcal{B}\hat{z}$ and a conducting cross bar $\ell\hat{x}$ at $y\hat{y}$ sliding on parallel conductors that terminate at a fixed transverse conductor, as in Fig. 6.4. Kick the cross bar so its initial velocity is $v_0\hat{y}$. Assume the initial speed v_0 is small enough that the induced current's magnetic field contributes negligibly to the magnetic flux though the loop. If the area element $d\vec{a} = da\hat{z}$, then the magnetic flux through the loop is

$$\Phi_{\mathcal{B}} = \iint_a \vec{\mathcal{B}} \cdot d\vec{a} = \iint_a \mathcal{B} \, da = \mathcal{B}(\ell y) > 0. \tag{6.21}$$

and it is increasing

$$\frac{d\Phi_{\mathcal{B}}}{dt} = \dot{\Phi}_{\mathcal{B}} = \mathcal{B}\ell\dot{y} = \mathcal{B}\ell v_y > 0. \tag{6.22}$$

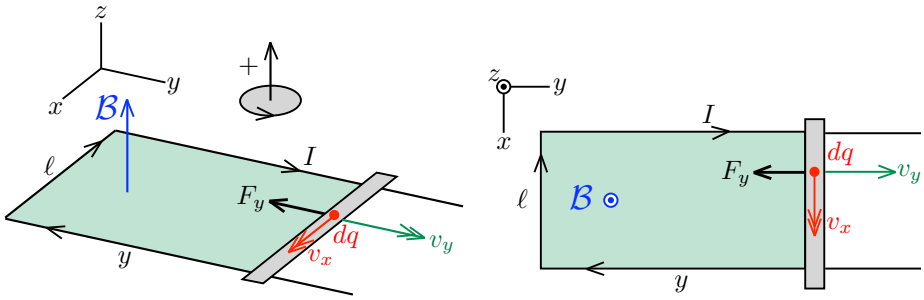


Figure 6.4: From a kicked start, the sliding cross bar of this linear electric generator exponentially slows, dissipating energy as heat in the rails.

The changing magnetic flux induces a circulating electric field that drives current around the loop formed by the cross bar, parallel conductors, like a delocalized battery. If the circuit is an **ohmic device** of resistance $\mathcal{R}$,

$$\mathit{\Gamma}_{\mathcal{E}} = I\mathcal{R}, \tag{6.23}$$

where a right-hand-rule relates the positive sense of circulation around the loop to the positive direction of the area bounded by the loop. Substitute circulation and rate of change of flux with time into the Eq. 6.12 Faraday's law to get

$$I\mathcal{R} = -\mathcal{B}\ell v_y \tag{6.24}$$

or

$$I = \frac{\mathcal{B}\ell}{\mathcal{R}} v_y < 0. \tag{6.25}$$

The induced current (clockwise from above) generates an induced magnetic field (downward through the loop) that opposes the change in the magnetic flux (increasing upward). Sometimes referred to as **Lenz's rule**, this electromagnetic "inertia" ensures energy conservation and is embodied mathematically in the minus sign in Faraday's law.

The constant magnetic field deflects the induced current in the cross bar with a force $F_y < 0$ that exponentially slows the motion. The sliding bar dissipates its kinetic energy as heat in the rails, and the total energy dissipated is the initial kinetic energy imparted by the kick. ■

6.3 Monopole and Dipole Detectors

If a magnetic charge passes through a loop of copper wire of resistance $\mathcal{R}$, then be Eq. 6.10, its circulating electric field will induce a circulating current

$$I = \frac{1}{\mathcal{R}}\Gamma_{\mathcal{E}} = -\frac{1}{\mathcal{R}}\dot{\Phi}_{\mathcal{B}} = \mu_0 \frac{Q_{\mathcal{B}}}{2\mathcal{R}} \frac{vr^2}{(z^2+r^2)^{3/2}}. \tag{6.26}$$

A bar magnet with a conventional north and south pole is a magnetic dipole. Model it by two magnetic monopoles $\pm Q_{\mathcal{B}}$ a distance d apart. If a bar magnet passes through a loop of copper wire, its circulating electric field will induce a circulating current

$$I = \mu_0 \frac{vr^2 Q_{\mathcal{B}}}{2\mathcal{R}} \left(\frac{1}{((z+d/2)^2+r^2)^{3/2}} - \frac{1}{((z-d/2)^2+r^2)^{3/2}} \right). \tag{6.27}$$

In both cases, the peak currents are proportional to the speed v, as in Fig. 6.5.

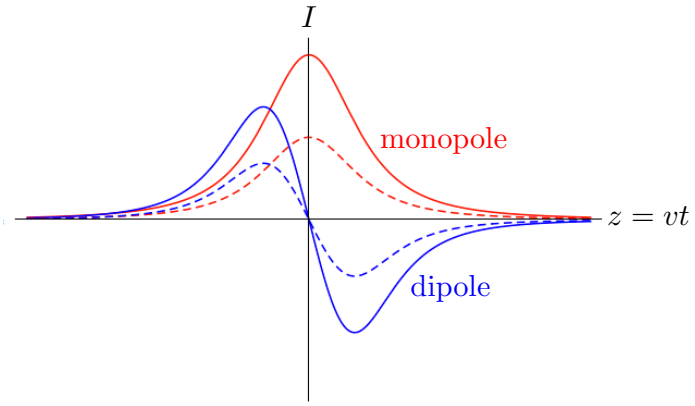


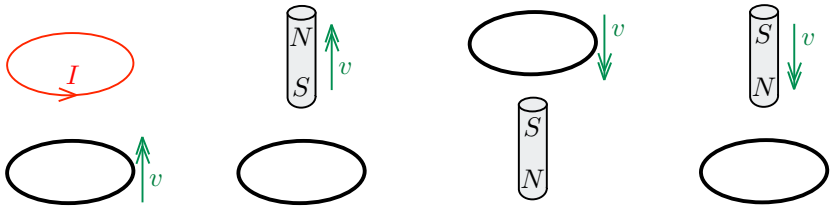
Figure 6.5: Current in a loop of wire as a magnetic monopole (red) and dipole (blue) pass through slowly (dashed) and quickly (solid).

6.4 Problems

1. Current loops (represented by thick, black circles) move relative to fixed current loops or bar magnets.

   (a) What are the directions of the currents induced in the loops?

   (b) What are the corresponding forces on the loops? (Hint: Exploit the Fig. 4.8 analogy between current loops and dipole bar magnets.)



2. Withdraw a horizontal, rectangular wire loop, of electrical resistance $\mathcal{R}$ and width w, lengthwise from a finite region of vertical magnetic field $\vec{\mathcal{B}}$ at a speed v. What is the magnitude and direction of the current induced in the wire?

3. A cylindrical volume of radius R confines a uniform magnetic field $\vec{\mathcal{B}} = kt^2\hat{z}$ that increases quadratically with time.

   (a) Use Faraday's law to find the (magnitude and direction of the) induced electric field $\vec{\mathcal{E}}$ everywhere, both inside and outside the cylindrical volume.

   (b) Qualitatively graph the induced electric field magnitude $\mathcal{E}$ as a function of the perpendicular separation s from the cylinder's axis.

4. Graph the flux of magnetic field through a circular loop as a magnetic charge passes through it. What happens at $z = 0$? (Hint: Equation 6.7 works only for $z > 0$.)

5. A long current with a U-shaped, hairpin curve carries a current I.

   (a) What is the magnetic field $\vec{\mathcal{B}}_a$ at the center of the hairpin curve?

   (b) What is the magnetic field $\vec{\mathcal{B}}_b$ midway between the straight segments far from the hairpin curve? (Hint: Exploit symmetry.)



Chapter 7

Maxwell

Maxwell's law relates a dynamic electric field to an induced magnetic field, and thereby completes the four Maxwell's equations that summarize electromagnetism.

7.1 Electric Flux and Magnetic Circulation

As a complement to the Section 6.1 derivation of Faraday's law, consider an electric charge incident on an imaginary loop. As it approaches the loop, more and more of its electric flux pierces the loop. Meanwhile, the electric charge's circulating magnetic field grows stronger and stronger at the loop. Calculate the rate of change with time of its electric flux through the loop and compare to its circulating magnetic field at the loop.

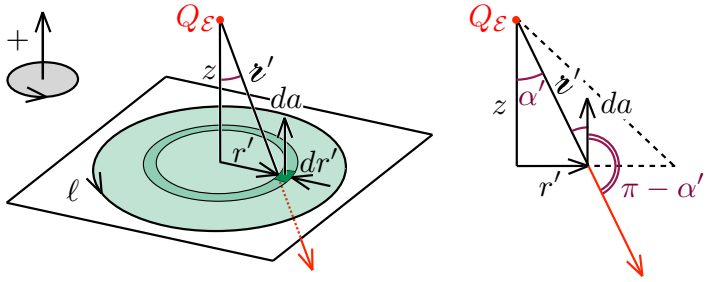


Figure 7.1: Geometry for electric charge flux calculation.

To calculate the flux, assume the electric charge $Q_\mathcal{E} > 0$ at a height $z > 0$ on the axis of a circular wire loop of radius r, as in Fig. 7.1. If up is positive so that the sum of the area elements $d\vec{a} = da\,\hat{z}$ at constant radii $r' < r$ are infinitesimal concentric rings of area $\int da = 2\pi r' dr'$, then the electric flux through the loop

is

$$\begin{aligned}\Phi_{\mathcal{E}} &= \iint_a \vec{\mathcal{E}} \cdot d\vec{a} \\ &= \iint_a \mathcal{E}\, da \cos[\pi - \alpha'] \\ &= \int_0^r \epsilon_0^{-1} \frac{Q_{\mathcal{E}}}{4\pi \boldsymbol{\imath}'^2} (2\pi r' dr')(-\cos\alpha') \\ &= -\epsilon_0^{-1} \frac{Q_{\mathcal{E}}}{2} \int_0^r \frac{\cos\alpha'\, r' dr'}{\boldsymbol{\imath}'^2}. \end{aligned} \tag{7.1}$$

Eliminate the variables r' and $\boldsymbol{\imath}'$ in favor of the angle α' using

$$\boldsymbol{\imath}' = z \sec\alpha' \tag{7.2}$$

and

$$r' = z \tan\alpha', \tag{7.3}$$

where α' ranges from 0 to α as r' ranges from 0 to r. Differentiate to find

$$dr' = z(\sec\alpha')^2\, d\alpha'. \tag{7.4}$$

Substitute to get

$$\begin{aligned}\Phi_{\mathcal{E}} &= -\epsilon_0^{-1} \frac{Q_{\mathcal{E}}}{2} \int_0^\alpha \frac{\cos\alpha'(z\tan\alpha')z(\sec\alpha')^2 d\alpha'}{(z\sec\alpha')^2} \\ &= -\epsilon_0^{-1} \frac{Q_{\mathcal{E}}}{2} \int_0^\alpha \sin\alpha'\, d\alpha' \\ &= -\epsilon_0^{-1} \frac{Q_{\mathcal{E}}}{2} (-\cos\alpha')\Big|_0^\alpha \\ &= -\epsilon_0^{-1} \frac{Q_{\mathcal{E}}}{2} (1 - \cos\alpha) < 0. \end{aligned} \tag{7.5}$$

If the electric charge moves downward at a speed $\dot{z} = -v < 0$, differentiate

$$\cot\alpha = \frac{z}{r} \tag{7.6}$$

with respect to time to get

$$-(\csc\alpha)^2 \dot{\alpha} = \frac{\dot{z}}{r} = -\frac{v}{r} \tag{7.7}$$

and solve to find the rate of change with time of the angle

$$\dot{\alpha} = -\frac{v}{r}(\sin\alpha)^2 = \frac{vr}{\boldsymbol{\imath}^2}. \tag{7.8}$$

Hence, electric flux through the loop changes with time at the rate

$$\begin{aligned}\dot{\Phi}_{\mathcal{B}} = \frac{d\Phi_{\mathcal{B}}}{dt} &= -\epsilon_0^{-1}\frac{Q_{\mathcal{E}}}{2}(0+\sin\alpha\,\dot{\alpha}) \\ &= -\epsilon_0^{-1}\frac{Q_{\mathcal{E}}}{2}\left(\frac{r}{\mathscr{r}}\right)\left(\frac{vr}{\mathscr{r}^2}\right). \\ &= -\epsilon_0^{-1}\frac{Q_{\mathcal{B}}}{2}\frac{vr^2}{(z^2+r^2)^{3/2}} < 0. \end{aligned} \tag{7.9}$$

The flux is negative and becoming more negative, decreasing in value but increasing in magnitude, as the charge approaches the loop from above.

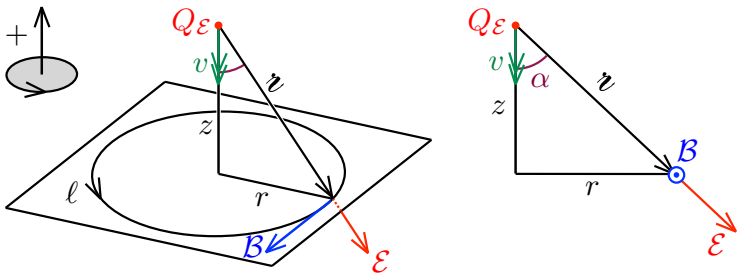


Figure 7.2: Geometry for electric charge circulation calculation.

Meanwhile, the moving charge's Eq. 4.4 magnetic field

$$\vec{\mathcal{B}} \approx \mu_0\epsilon_0\vec{v}\times\vec{\mathcal{E}} \approx \mu_0\vec{v}\times\frac{Q_{\mathcal{E}}}{4\pi\mathscr{r}^2}\hat{\mathscr{r}} \tag{7.10}$$

produces in the loop the the magnetic circulation

$$\begin{aligned}\Gamma_{\mathcal{B}} &= \oint_\ell \vec{\mathcal{B}}\cdot d\vec{\ell} \\ &= \mathcal{B}\oint_\ell d\ell\cos\pi \\ &\approx -v\left(\mu_0\frac{Q_{\mathcal{E}}}{4\pi\mathscr{r}^2}\right)\sin\alpha\oint_\ell d\ell \\ &= -v\left(\mu_0\frac{Q_{\mathcal{E}}}{4\pi\mathscr{r}^2}\right)\left(\frac{r}{\mathscr{r}}\right)(2\pi r) \\ &= -\mu_0\frac{Q_{\mathcal{E}}}{2}\frac{vr^2}{\mathscr{r}^3} \\ &= -\mu_0\frac{Q_{\mathcal{E}}}{2}\frac{vr^2}{(z^2+r^2)^{3/2}} < 0. \end{aligned} \tag{7.11}$$

Like the flux, the circulation is negative and becoming more negative, decreasing in value but increasing in magnitude, as the charge approaches the loop from

above. Indeed, by Eq. 7.9, the circulation is indeed proportional to the rate of change of the magnetic flux,

$$\Gamma_{\mathcal{B}} = +\epsilon_0\mu_0\dot{\Phi}_{\mathcal{E}}. \tag{7.12}$$

7.2 Maxwell's Law

More generally, the circulation of the magnetic field around *any* closed loop ℓ is minus the rate of change with time of the electric flux through *any* area a bounded by the loop,

$$\oint_l \vec{\mathcal{B}} \cdot d\vec{\ell} = \Gamma_{\mathcal{B}} = \epsilon_0\mu_0\dot{\Phi}_{\mathcal{E}} = \epsilon_0\mu_0\frac{d\Phi_{\mathcal{E}}}{dt} = \epsilon_0\mu_0\frac{d}{dt}\iint_a \vec{\mathcal{E}} \cdot d\vec{a}, \tag{7.13}$$

which is **Maxwell's law**. Unlike Faraday, who induced his law from many experiments, Maxwell postulated his law to symmetrize the equations of electromagnetism.

7.2.1 Example: Charging Capacitor

The two circular electrodes of a parallel plate capacitor produce a uniform but increasing magnetic field magnitude $\vec{\mathcal{E}}[t] = (\mathcal{E}_0 + kt)\hat{z}$, where the rate $k > 0$, as in Fig. 7.3.

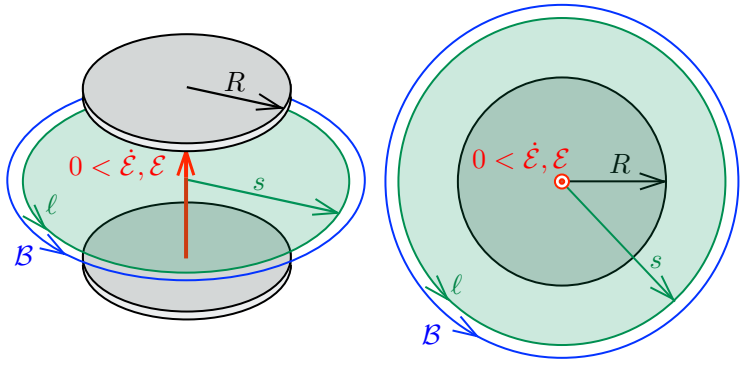


Figure 7.3: The time varying electric field $\vec{\mathcal{E}}[t]$ of a charging capacitor induces a circulating magnetic field $\vec{\mathcal{E}}[t]$.

By symmetry, the induced magnetic field $\vec{\mathcal{B}}[\vec{r}] = \mathcal{B}_\phi[s]\hat{\phi}$ circulates around the cylinder. While Maxwell's law works for any loop, a circular loop of radius s concentric with the cylinder mirrors the symmetry of the problem. The magnetic

circulation around the loop

$$\Gamma_{\mathcal{B}} = \oint_{\text{circle}} \vec{\mathcal{B}} \cdot d\vec{\ell} = \mathcal{B}_\phi[s] \oint_{s=\text{const}} d\ell = \mathcal{B}_\phi \, 2\pi s.$$

If the Maxwellian loop is outside the cylinder, $s > R$, the electric flux through the loop

$$\Phi_{\mathcal{E}} = \iint_{\text{disk}} \vec{\mathcal{E}} \cdot d\vec{a} = (\pi R^2)(\mathcal{E}_0 + kt), \tag{7.14}$$

and its rate of change with time

$$\dot{\Phi}_{\mathcal{E}} = \frac{d\Phi_{\mathcal{E}}}{dt} = \pi R^2 k. \tag{7.15}$$

Substitute circulation and rate of change of flux into the Eq. 7.12 Maxwell law to get

$$\mathcal{B}_\phi \, 2\pi s = \epsilon_0 \mu_0 \pi R^2 k \tag{7.16}$$

and conclude

$$\mathcal{B}_\phi = \epsilon_0 \mu_0 \frac{1}{2} k \frac{R^2}{s}, \tag{7.17}$$

so the magnetic field decreases inversely with separation s from the axis.

Alternately, if the Maxwellian loop is inside the cylinder, $s < R$, the electric flux through the loop

$$\Phi_{\mathcal{E}} = \iint_{\text{disk}} \vec{\mathcal{E}} \cdot d\vec{a} = (\pi s^2)(\mathcal{E}_0 + kt), \tag{7.18}$$

and its rate of change with time

$$\dot{\Phi}_{\mathcal{E}} = \frac{d\Phi_{\mathcal{E}}}{dt} = \pi s^2 k. \tag{7.19}$$

Equate circulation and minus rate of change of flux

$$\mathcal{B}_\phi \, 2\pi s = \epsilon_0 \mu_0 \pi s^2 k. \tag{7.20}$$

and conclude

$$\mathcal{B}_\phi = \epsilon_0 \mu_0 \frac{1}{2} k s > 0, \tag{7.21}$$

so the magnetic field increases proportionally with separation s from the axis, counterclockwise from above in Fig. 7.3. ■

7.3 Maxwell's Equations

Specifying the flux and circulation of a vector field everywhere (through all areas and around all loops) completely characterizes the vector field. Working after Coulomb, Gauss, Biot-Savart, Ampère, and Faraday, and combining their insights with his own, Maxwell did this for the electromagnetic field in the 1860s.

Combining the Ampère and Maxwell laws into a single Maxwell-Ampère law

$$\Gamma_{\mathcal{B}} = +\epsilon_0\mu_0\dot{\Phi}_{\mathcal{E}} + \mu_0 I_{\mathcal{E}}, \tag{7.22}$$

Tables 7.1 and 7.2 summarize Maxwell's equations with and without magnetic monopoles.

Table 7.1: Maxwell's equations with electric and magnetic charges $Q_{\mathcal{E}}$ and $Q_{\mathcal{B}}$.

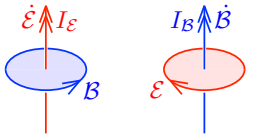
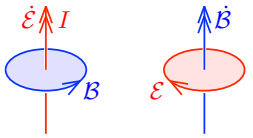
$\epsilon_0\Phi_{\mathcal{E}} = Q_{\mathcal{E}}$		$\Phi_{\mathcal{B}} = \mu_0 Q_{\mathcal{B}}$
$\Gamma_{\mathcal{B}} = +\epsilon_0\mu_0\dot{\Phi}_{\mathcal{E}} + \mu_0 I_{\mathcal{E}}$		$\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}} - \mu_0 I_{\mathcal{B}}$

Table 7.2: Maxwell's equations with electric charges and currents Q and $\dot{Q} = I$.

$\epsilon_0\Phi_{\mathcal{E}} = Q$		$\Phi_{\mathcal{B}} = 0$
$\Gamma_{\mathcal{B}} = +\epsilon_0\mu_0\dot{\Phi}_{\mathcal{E}} + \mu_0 I$		$\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}$

Gauss's law for the electric field, $\epsilon_0\Phi_{\mathcal{E}} = Q$, means that the flux of the electric field through any closed surface is the electric charge inside. **Gauss's law** for the magnetic field, $\Phi_{\mathcal{B}} = 0$, means that the flux of the magnetic field through any closed surface vanishes, so whatever goes in must come out. **Faraday's law**, $\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}$, means that the circulation of the electric field around any closed loop is *minus* the time rate of change of the magnetic flux through any surface bounded by the loop. **Maxwell-Ampère's law**, $\Gamma_{\mathcal{B}} = +\epsilon_0\mu_0\dot{\Phi}_{\mathcal{E}} + \mu_0 I$, means that the circulation of the magnetic field around any closed loop is the time rate of change of the electric flux through any surface bounded by the loop *plus* the electric current passing through the loop. (The special, stationary case of Ampère's original law, $\Gamma_{\mathcal{B}} = \mu_0 I$, means the circulation of the magnetic field around any closed loop is the electric current passing though the loop.)

While charge and current sources determine the electric and magnetic fields via Maxwell's equations, the fields determine forces by the Eq. 4.6 Lorentz force law. Forces determine motion by Newton second law and its relativistic generalization.

In his famous lectures on physics, Richard Feynman wrote [4], "From a long view of the history of mankind – seen from, say, ten thousand years from now, there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade."

7.4 Electromagnetic Radiation

Maxwell's equation in a vacuum, with no charges or currents, contain a revelation: electromagnetic radiation. The vanishing of the fluxes, $\Phi_{\mathcal{E}} = 0$ and $\Phi_{\mathcal{B}} = 0$, imply that the electric and magnetic fields can't start and stop but must circulate. Circulating fields are threaded by time varying fluxes, $\mathit{\Gamma}_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{E}}$ and $\mathit{\Gamma}_{\mathcal{B}} = +\dot{\Phi}_{\mathcal{B}}\epsilon_0\mu_0$, so that a changing electric flux induces a circulating magnetic field, and a changing magnetic flux induces a (counter) circulating electric field. There exists the possilbility of a self-regenerating **electromagnetic wave**, an endless dance of electric and magnetic fields.

7.4.1 Example: Kicked Charge Sheet

From Example 3.2.2, an xz-plane with area charge density $\sigma > 0$ has a perpendicular electric field $\vec{\mathcal{E}}_0 = \pm\hat{y}\sigma/2$. From Example 5.2.2, an xz-plane with linear current density $\vec{K} = \hat{x}\,K$ has a parallel magnetic field $\vec{\mathcal{B}}_0 = \pm\hat{z}K/2$, where $K > 0$. Relate the two by kicking the first. At time $t = 0$, suddenly set the plane in motion parallel to itself with velocity $\vec{v} = \hat{x}\,v$. Thereby induce *dynamic* electric and magnetic fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ that propagate outward at speed c. Find this speed by applying Maxwell's circulation equations to closed loops enclosing the wavefront, as in Fig. 7.4.

At the wavefront, the electric field can't just disappear. If it did, the change in the electric field would induce a magnetic field, which in turn would induce an electric field, and so on. Apply Faraday's law

$$\mathit{\Gamma}_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}} \tag{7.23}$$

to a rectangular loop at a distance y_0 from the charge sheet in the xy-plane straddling the wavefront to get

$$\mathcal{E}\Delta x + 0 + 0 + 0 = -\frac{d}{dt}\Big(- \mathcal{B}(ct - y_0)\Delta x\Big), \tag{7.24}$$

or

$$\mathcal{E} = \mathcal{B}c. \tag{7.25}$$

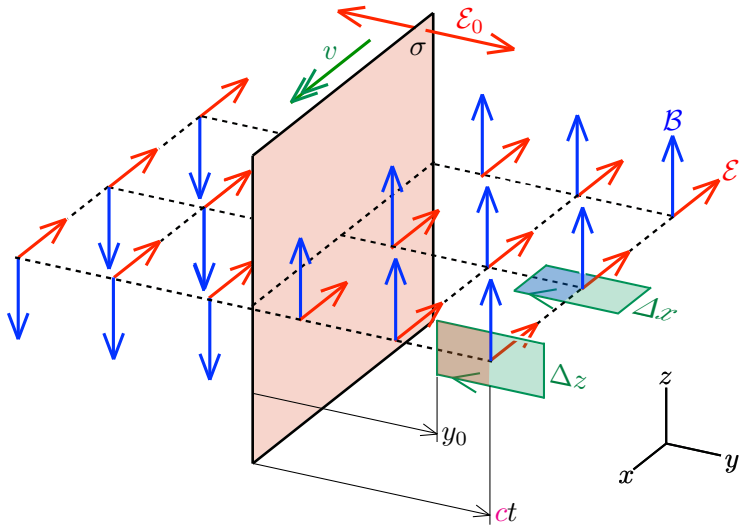


Figure 7.4: Electromagnetic fields propagate a distance ct from a charge sheet suddenly set in motion at time $t = 0$.

Similarly, at the wavefront, the magnetic field can't just disappear. If it did, the change in the magnetic field would induce an electric field, which in turn would induce a magnetic field, and so on. Apply Maxwell's law

$$\Gamma_{\mathcal{B}} = +\dot{\Phi}_{\mathcal{E}}\,\epsilon_0\mu_0 \tag{7.26}$$

to a rectangular loop at a distance y_0 from the charge sheet in the yz-plane straddling the wavefront to get

$$\mathcal{B}\Delta z + 0 + 0 + 0 = +\frac{d}{dt}\Big(+ \mathcal{E}(ct - y_0)\Delta z\Big)\epsilon_0\mu_0, \tag{7.27}$$

or

$$\mathcal{B} = \mathcal{E}c\,\epsilon_0\mu_0. \tag{7.28}$$

Multiply the Eq. 7.25 electric field by the Eq. 7.28 magnetic field to find

$$\mathcal{E}\mathcal{B} = \mathcal{B}\mathcal{E}\,c^2\epsilon_0\mu_0, \tag{7.29}$$

and solve for the wavefront speed

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}, \tag{7.30}$$

which is the speed of light, as advertised in Section 1.2. Furthermore, the propagating electric and magnetic fields are perpendicular to each other and to the direction of propagation, forming the right-handed set

$$\hat{\mathcal{E}} \times \hat{\mathcal{B}} = \hat{c}, \tag{7.31}$$

which is a generic feature of electromagnetic waves.

From Example 5.2.2, the electric field of the kicked sheet is proportional to the speed

$$\mathcal{E} = \mathcal{B}c = \mu_0 \frac{K}{2} c = \mu_0 \frac{\sigma v}{2} c \tag{7.32}$$

but oppositely directed

$$\epsilon_0 \vec{\mathcal{E}} = -\frac{1}{2}\sigma \frac{\vec{v}}{c}. \tag{7.33}$$

■

7.4.2 Example: Decelerated Point Charge

A point charge Q moving at velocity $\vec{v} = \hat{z}\, v$ slows at time $t = 0$ and position $z = 0$ to rapidly stop at time $t = \tau$ and position $z = v\tau/2$ with acceleration $\vec{a} = -\hat{z}\, v/\tau$, as in Fig. 7.5.

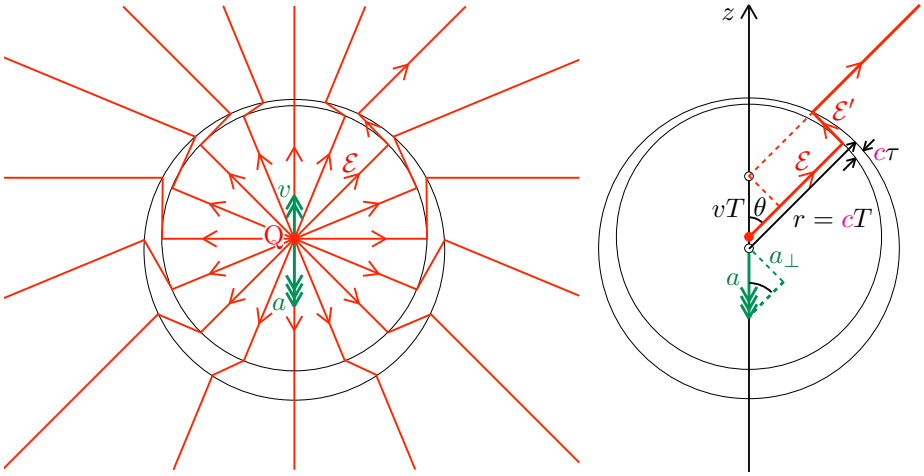


Figure 7.5: Deceleration of a point charge generates a kink in its electric field lines that propagates outward at light speed. For rapid decelerations, the spheres that bound the kink (shown in cross sections) are concentric with the origin.

At time $t < 0$, the charge carries its radial electric field with it. At time $t > \tau$, its far fields have not yet "got the news" that the charge has stopped, so the electric field is radially away from where it would have been if the charge

had not stopped. More specifically, at time $t = T \gg \tau$, for far observers the field is that of a charge at position $z = vT$, and for near observers the field is of a charge at rest at position $z = 0$. In a shell of characteristic thickness $(c \pm v)\tau \approx c\tau$ for $v \ll c$, the electric field lines kink to join the near and far fields.

From Fig. 7.5, the ratio of the tangential and radial components of the electric field in the kink is

$$\frac{\mathcal{E}'_\theta}{\mathcal{E}'_r} \approx -\frac{vT\sin\theta}{c\tau} < 0. \tag{7.34}$$

If a tiny Gaussian cylinder of small cross sectional area a and vanishing height $h \to 0$ radially straddles the inner surface of the kink, then Gauss's law $\Phi_\mathcal{E} = \epsilon_0^{-1} Q_{\text{in}}$ implies

$$-\mathcal{E}_r a + 0 + \mathcal{E}'_r a = 0, \tag{7.35}$$

so the radial component of the field is continuous,

$$\mathcal{E}'_r = \mathcal{E}_r \approx \epsilon_0^{-1}\frac{Q}{4\pi r^2} \approx \epsilon_0^{-1}\frac{Q}{4\pi c^2 T^2} = \mu_0 \frac{Q}{4\pi T^2}, \tag{7.36}$$

and the tangential component

$$\begin{aligned}
\mathcal{E}'_\theta &= -\frac{vT\sin\theta}{c\tau}\mathcal{E}'_r \\
&\approx -\frac{vT\sin\theta}{c\tau}\left(\mu_0\frac{Q}{4\pi T^2}\right) \\
&= -\mu_0\frac{Q}{4\pi(cT)}\left(\frac{v}{\tau}\right)\sin\theta \\
&= -\mu_0\frac{Q}{4\pi r}a\sin\theta \\
&= -\mu_0\frac{Q}{4\pi r}a_\perp.
\end{aligned} \tag{7.37}$$

More generally, if the charge is at a position $\vec{\boldsymbol{\imath}}$ relative to the observer, then the radiation electric field is proportional to the projection of the acceleration transverse to the line of sight

$$\vec{\mathcal{E}}_{\text{rad}} \approx -\mu_0\frac{Q}{4\pi\boldsymbol{\imath}}\vec{a}_\perp, \tag{7.38}$$

which is **Feynman's formula**. The radiation field decreases as the inverse of the distance $1/\boldsymbol{\imath}$, while the Coulomb field decrease as the inverse *square* of the distance $1/\boldsymbol{\imath}^2$. Consequently, far from the source the radiation field dominates. The radiation field is maximum perpendicular to the acceleration and vanishes parallel to the acceleration, as in Fig. 7.6. There is also a magnetic field $\mathcal{B}_{\text{rad}}$ in the kink and, as in Example 7.4.1, it forms a right-handed set with the electric field and the propagation velocity, $\hat{\mathcal{E}}_{\text{rad}} \times \hat{\mathcal{B}}_{\text{rad}} = \hat{c}$.

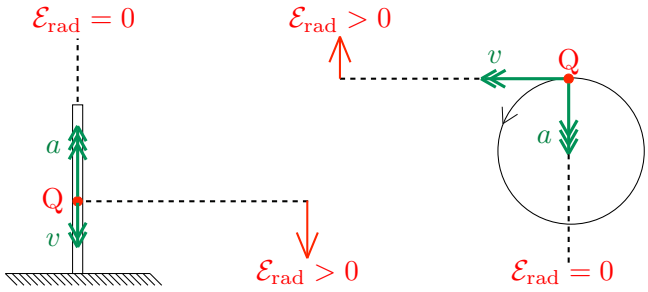


Figure 7.6: Radiations from an oscillating charge in a radio antenna (left) and a charge revolving in a circle (right) are proportional to the projection of the acceleration on the line of sight.

Because kinks in the field need time to propagate from the source to the observation point, the radiation field produced by an acceleration $\vec{a}$ at time t reaches distance r at a time r/c later,

$$\vec{\mathcal{E}}_{\rm rad}\left[t+\frac{r}{c}\right] \approx -\mu_0 \frac{Q}{4\pi r}\vec{a}_\perp[t] \tag{7.39}$$

or

$$\vec{\mathcal{E}}_{\rm rad}\,[t] \approx -\mu_0 \frac{Q}{4\pi r}\vec{a}_\perp\left[t-\frac{r}{c}\right] \approx -\mu_0 \frac{Q}{4\pi r}\vec{a}_\perp\,[t_r]\,, \tag{7.40}$$

where

$$t_r = t - \frac{r}{c} \tag{7.41}$$

is the *retarded time*. Feynman's formula is exact if the charge is momentarily at rest and is a good approximation if its speed is nonrelativistic $v \ll c$. ■

7.5 Problems

1. For Fig. 7.4, show that the induced electric field $\vec{\mathcal{E}}$ on the far side $y < 0$ of the kicked charge sheet is drawn correctly.

2. A cylindrical volume of radius R confines a uniform electric field $\vec{\mathcal{B}} = kt^3\hat{z}$ that increases cubically with time.

   (a) Use Maxwell's law to find the (magnitude and direction of the) induced electric field $\vec{\mathcal{B}}$ everywhere, both inside and outside the cylindrical volume.

   (b) Qualitatively graph the induced electric field magnitude $\mathcal{B}$ as a function of the perpendicular separation s from the cylinder's axis.

3. Draw a diagram analogous to Fig. 7.5 showing the kinked electric field lines for a charge that begins at rest, is accelerated to motion, and then is decelerated back to rest.

Part II

Applications

Chapter 8

Electromagnetic Waves

Light is a transverse electromagnetic wave that can propagate in a vacuum or in matter.

8.1 Travelling Waves

Pump one end of a horizontal string up and down once to generate a pulse that propagates along the string at constant velocity $\vec{v} = v\hat{y}$, as in Fig. 8.1.

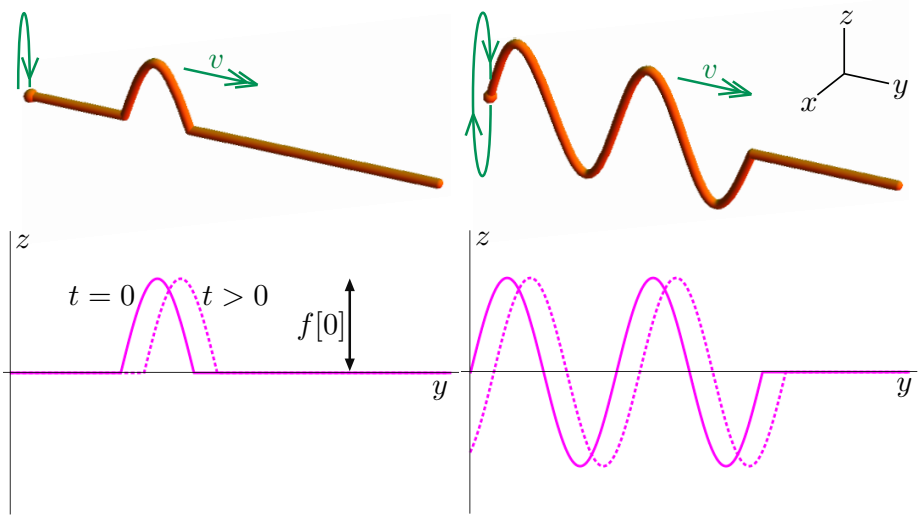


Figure 8.1: Wave pulse and wave train on a pumped string (top) and the corresponding idealized graphs (bottom).

If $z[y,t]$ is the height of the pulse at position y and time t, then initially

$$z[y,0] = f[y] \tag{8.1}$$

and subsequently

$$z[y,t] = f[y - vt], \tag{8.2}$$

because the peak $f[0]$ is always at $x - vt = 0$ so $x = vt$.

Pump one end of the horizontal string up and down cyclicly in time T to generate successive peaks (and troughs) separated by distance λ that propagate along the string at constant velocity $\vec{v} = v\hat{y}$, also as in Fig. 8.1. If $z[y,t]$ is the height of the wave train at position y and time t, then initially

$$z[y,0] = A\sin kz \tag{8.3}$$

and subsequently

$$z[y,t] = A\sin[k(y-vt)] = A\sin[ky - \omega t] = A\sin\left[2\pi\left(\frac{y}{\lambda} - \frac{t}{T}\right)\right], \tag{8.4}$$

where the spatial frequency or **wave number**

$$k = \frac{2\pi}{\lambda}, \tag{8.5}$$

and the temporal or **angular frequency**

$$\omega = \frac{2\pi}{T}. \tag{8.6}$$

Historically and for convenience, distinguish angular frequency ω with SI unit of **radians per second**,

$$\text{unit}[\omega] = \frac{\text{rad}}{\text{s}}, \tag{8.7}$$

from **frequency**

$$\nu = \frac{\omega}{2\pi} = \frac{1}{T} \tag{8.8}$$

with SI unit of **hertz**,

$$\text{unit}[\nu] = \frac{1}{\text{s}} = \text{Hz}. \tag{8.9}$$

The sinusoidal wave train travels one wavelength λ in one period T at speed

$$v = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k} \tag{8.10}$$

or

$$v = \lambda\nu. \tag{8.11}$$

8.2 Transverse Electromagnetic Wave

Assume a transverse, sinusoidal electromagnetic wave oscillating or **polarized** in the z-direction and traveling in the the y-direction with fields

$$\vec{\mathcal{E}}[y,t] = \hat{z}\,\mathcal{E}_m \sin[ky - \omega t], \tag{8.12a}$$

$$\vec{\mathcal{B}}[y,t] = \hat{x}\,\mathcal{B}_m \sin[ky - \omega t], \tag{8.12b}$$

as in Fig. 8.2. Check that these dynamics fields satisfy Maxwell's equations.

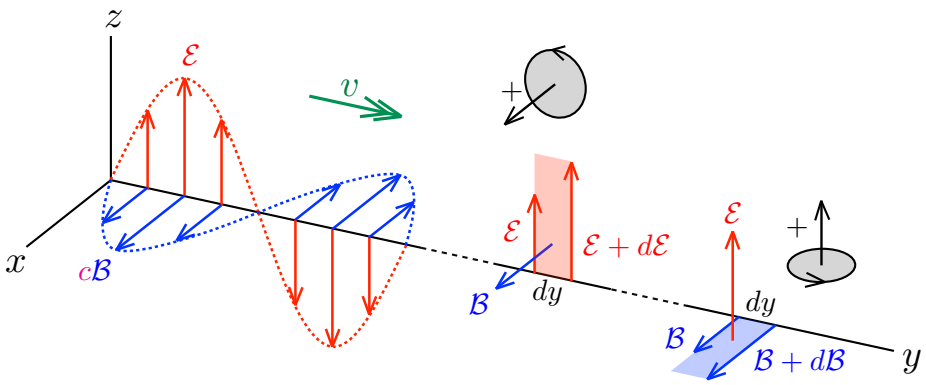


Figure 8.2: Changing magnetic flux induces a circulating electric field and changing electric flux induces a (counter) circulating magnetic field in a transverse electromagnetic wave.

In the yz-plane, a changing magnetic flux induces a circulating electric field by Faraday's law

$$\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}, \tag{8.13}$$

so

$$(\mathcal{E}_z + d\mathcal{E}_z)\Delta z + 0 - \mathcal{E}_z \Delta z + 0 = -\frac{d}{dt}\big(+ \mathcal{B}_x \Delta z\, dy\big) \tag{8.14}$$

or

$$\frac{d\mathcal{E}_z}{dy} = -\frac{d\mathcal{B}_x}{dt} \tag{8.15}$$

or better

$$\frac{\partial \mathcal{E}_z}{\partial y} = -\frac{\partial \mathcal{B}_x}{\partial t}, \tag{8.16}$$

where the partial derivative notation ∂, pronounced "del", is a reminder to differentiate with respect to one variable while holding the others constant. Substitute the Eq. 8.12 fields to get

$$k\mathcal{E}_m \cos[ky - \omega t] = +\omega \mathcal{B}_m \cos[ky - \omega t], \tag{8.17}$$

which is true if

$$\frac{\mathcal{E}_m}{\mathcal{B}_m} = \frac{\omega}{k}. \tag{8.18}$$

Meanwhile, in the xy-plane, a changing electric flux induces a (counter) circulating magnetic field by Maxwell's law

$$\Gamma_{\mathcal{B}} = +\dot{\Phi}_{\mathcal{E}}\epsilon_0\mu_0, \tag{8.19}$$

so

$$\mathcal{B}_x\Delta x + 0 - (\mathcal{B}_x + d\mathcal{B}_x)\Delta x + 0 = +\frac{d}{dt}\big(+\mathcal{E}_z\Delta x\, dy\big)\epsilon_0\mu_0 \tag{8.20}$$

or

$$-\frac{d\mathcal{B}_x}{dy} = \frac{d\mathcal{E}_z}{dt}\epsilon_0\mu_0 \tag{8.21}$$

or better

$$\frac{\partial\mathcal{B}_x}{\partial y} = -\frac{\partial\mathcal{E}_z}{\partial t}\epsilon_0\mu_0, \tag{8.22}$$

where again the partial derivative notation ∂ is a reminder to differentiate with respect to one variable while holding the others constant. Substitute the Eq. 8.12 fields to get

$$k\mathcal{B}_m\cos[ky-\omega t] = +\omega\mathcal{E}_m\cos[ky-\omega t]\epsilon_0\mu_0, \tag{8.23}$$

which is true provided

$$\frac{\mathcal{B}_m}{\mathcal{E}_m} = \frac{\omega}{k}\epsilon_0\mu_0. \tag{8.24}$$

Combine the Eq. 8.18 and Eq. 8.24 expressions for the ratio of the field amplitudes to get

$$\frac{k}{\omega} = \frac{\omega}{k}\epsilon_0\mu_0, \tag{8.25}$$

and solve for the wave speed

$$v = \frac{\lambda}{T} = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0\mu_0}} = c, \tag{8.26}$$

as in Example 7.4.1.

8.3 Electromagnetic Spectrum

The spectrum of electromagnetic radiation includes radio waves, microwaves, infrared (IR), visible, ultraviolet (UV), X-rays and γ-rays, as in Fig. 8.3, where frequency ν and wavelength λ are inversely related by

$$\lambda\nu = c \tag{8.27}$$

or

$$\nu = \frac{c}{\lambda} \approx \frac{300\text{m}/\mu\text{s}}{\lambda} = 300\text{MHz}\left(\frac{1\text{m}}{\lambda}\right). \tag{8.28}$$

The solar spectrum peaks in green, with wavelength

$$\lambda_g \approx \frac{1}{2}\mu\text{m} = 500\text{nm} = 5000\text{Å} \tag{8.29}$$

and frequency

$$\nu_g \approx 600\text{THz} = 600\,\text{million}\,\text{MHz}. \tag{8.30}$$

The color green is about 600 million MHz on a radio dial.

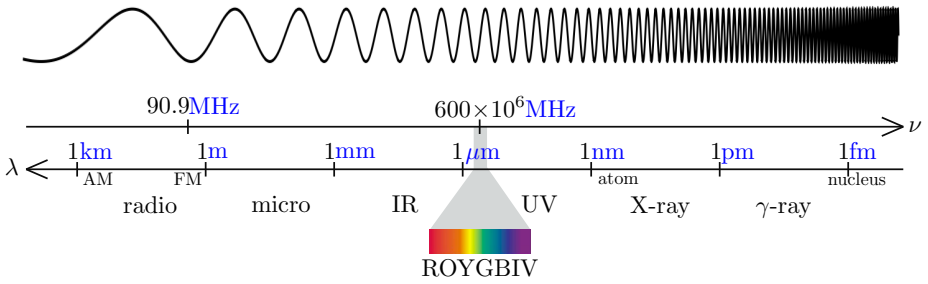


Figure 8.3: The electromagnetic spectrum or "Maxwell's rainbow" unifies disparate phenomena.

The natural lens of the human eye is opaque to ultraviolet light and indeed damaged by it. Some people with implanted artificial plastic lenses can see slightly into the ultraviolet. What does it look like? Purportedly, it's more violet than violet!

8.4 Slow Light

In matter such as glass or water, light slows to an *effective* speed

$$v = \frac{c}{n} < c, \tag{8.31}$$

where $n > 1$ is the matter's **index of refraction**. This has profound consequences, such as the **refraction** or bending of light in glass, which is exploited in the design of eyeglasses, microscopes, and telescopes. Since most of the mass of the matter is concentrated in the nuclei of its atoms, the matter is mostly empty space through which light travels at its vacuum speed c. However, the incident electromagnetic radiation dynamically **polarizes** the matter, as in Fig. 8.4, accelerating its electrons and thereby inducing them to radiate. The superposition of the incident field and the induced field is the *total* electromagnetic field

$$\vec{\mathcal{E}}_T = \vec{\mathcal{E}} + \vec{\mathcal{E}}', \tag{8.32}$$

which propagates slower than the incident field alone.

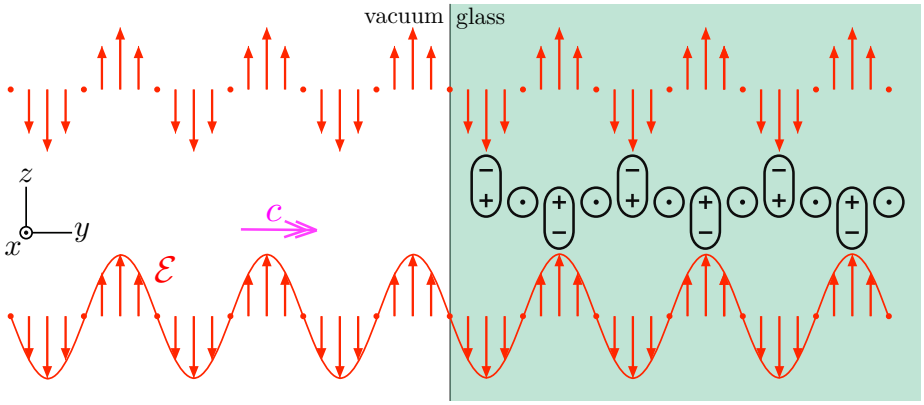


Figure 8.4: An electromagnetic wave induces a time varying polarization in the glass whose radiation modifies the incident field.

8.4.1 Example: Dielectric Refracting

Assume a sinusoidal electromagnetic wave polarized in the z-direction and traveling in the the y-direction with electric field

$$\vec{\mathcal{E}}[y,t] = \hat{z}\mathcal{E}_m \sin[ky - \omega t] \tag{8.33}$$

is incident on an xz atomic plane, as in Fig. 8.5. As in a **dielectric**, assume the plane's electrons are elastically bound to their atoms with effective restoring force per unit mass (or spring constant) κ. The equation of motion is

$$M_e \ddot{z}[t] = F_z[t] = -\kappa z[t] + Q_e \mathcal{E}_z[0,t] \tag{8.34}$$

or

$$M_e(\ddot{z}[t] + \omega_0^2 z[t]) = -Q_e \mathcal{E}_m \sin[\omega t], \tag{8.35}$$

where $\omega_0 = \sqrt{\kappa/M_e}$ is the resonant frequency of the electrons bound to their atoms. Substitute the particular sinusoidal solution

$$z[t] = A \sin[\omega t], \tag{8.36}$$

and solve for the amplitude A to get

$$z[t] = -\frac{Q_e}{M_e}\frac{1}{\omega_0^2 - \omega^2}\mathcal{E}_m \sin[\omega t]. \tag{8.37}$$

Differentiate to find the velocity

$$v_z[t] = \dot{z}[t] = -\frac{Q_e}{M_e}\frac{\omega}{\omega_0^2 - \omega^2}\mathcal{E}_m \cos[\omega t]. \tag{8.38}$$

If $t_r = t - y/c$ is the Eq. 7.41 retarded time, then from Eq. 7.33, the corresponding induced electric field of all the plane's electrons oscillating in unison is

$$\begin{aligned}\mathcal{E}'_z[y,t] &= -\frac{\sigma v_z[t_r]}{2}\mu_0 c \\ &= \frac{\sigma}{2}\frac{Q_e}{M_e}\frac{\omega}{\omega_0^2-\omega^2}\mathcal{E}_m \cos\left[\omega\left(t-\frac{y}{c}\right)\right]\mu_0 c \\ &= \mathcal{E}'_m \cos[ky-\omega t], \end{aligned} \tag{8.39}$$

where the induced amplitude

$$\mathcal{E}'_m = \mathcal{E}_m \frac{\sigma}{2}\frac{Q_e}{M_e}\frac{\omega}{\omega_0^2-\omega^2}\mu_0 c > 0, \tag{8.40}$$

provided the frequency $\omega < \omega_0$ is sub-resonant, as the case for visible light in glass, which has a resonance ω_0 in the ultraviolet.

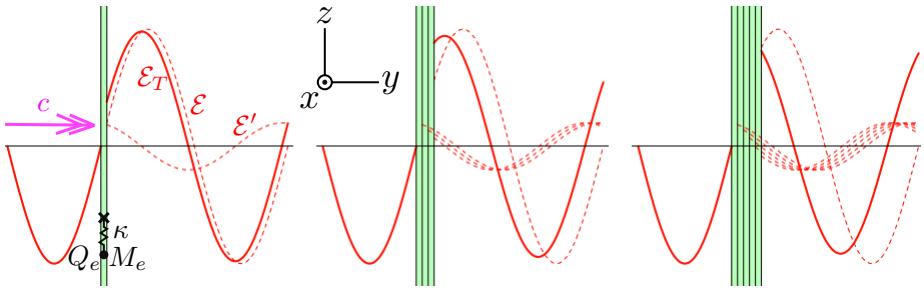


Figure 8.5: Superposition of incident and induced fields from a dielectric progressively delays the passage of an electromagnetic wave through successive atomic planes, thereby reducing its effective speed. (Not shown is a corresponding reflected wave.)

Since the Eq. 8.39 induced field $\vec{\mathcal{E}}'$ is shifted one quarter wavelength (or 90° out of phase) from the Eq. 8.33 incident field $\vec{\mathcal{E}}$, the Eq. 8.32 total field $\vec{\mathcal{E}}_T$ is effectively delayed. If $\mathcal{E}'_m/\mathcal{E}_m = \delta \ll 1$ so that $\sin\delta \approx \delta$ and $\cos\delta \approx 1$, then the total field

$$\begin{aligned}\mathcal{E}_{Tz} &= \mathcal{E}_z + \mathcal{E}'_z \\ &= \mathcal{E}_m \sin[ky-\omega t] + \mathcal{E}'_m \cos[ky-\omega t] \\ &= \mathcal{E}_m(\sin[ky-\omega t] + \delta\cos[ky-\omega t]) \\ &\approx \mathcal{E}_m(\sin[ky-\omega t]\cos\delta + \sin\delta\cos[ky-\omega t]) \\ &= \mathcal{E}_m \sin[ky-\omega t+\delta], \end{aligned} \tag{8.41}$$

which is shifted *backwards*. This delay progressively accumulates as the wave propagates through successive atomic planes, as in Fig. 8.5.This simple model

thereby accounts for the effective slowing of light in matter $v = c/n$ and the origin of the index of refraction n. ■

Typical refractive indices include $n_a \approx 1.3$ for air and $n_w \approx 1.5$ for glass. For certain frequencies ω, the refractive indices of an ionized gas or **plasma** can be less than unity, $n[\omega] < 1$, but plasmas still can't transmit *information* faster than light in a vacuum, $v > c$. *Negative* refractive indices are possible in certain **metamaterials**, which may lead to improved optical components like flat lenses. Light has been radically slowed and effectively "stopped" in exotic matter like ultra-cold **Bose-Einstein** condensates. The **slow glass** in Bob Shaw's famous science fiction short story "Light of other days" [5] delays the passage of light by years or more and enables the construction of "windows" that enable urban dwellers to enjoy landscape views.

8.4.2 Example: Conductor Reflecting

Unlike dielectrics like glass, which can be polarized by applied electric fields, **conductors** like copper contain sheets of mobile electrons of charge density σ. An incident electromagnetic wave induces an oscillating current of density $K = \sigma v$ that generates an electromagnetic wave that cancels the incident wave forward,

$$\mathcal{E}_{Tz}[y > 0, t] = \mathcal{E}_m \cos[ky - \omega t] - \mathcal{E}_m \cos[ky - \omega t] = 0, \tag{8.42}$$

and combines with the incident wave backward creating a **standing wave**,

$$\begin{aligned} \mathcal{E}_{Tz}[y < 0, t] &= \mathcal{E}_m \cos[ky - \omega t] - \mathcal{E}_m \cos[ky + \omega t] \\ &= 2\mathcal{E}_m \sin[ky] \sin[\omega t]. \end{aligned} \tag{8.43}$$

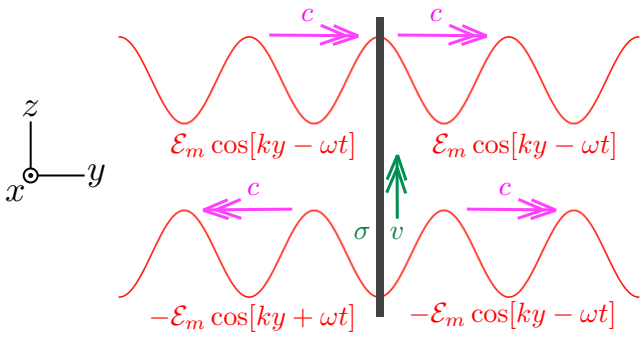


Figure 8.6: Superposition of incident and induced fields from a conducting sheet cancel forward and combine to a standing wave backward.

Perfect conductors are endless reservoirs of mobile electrons that provide the charge and current density to reflect any incident wave. In this limit, the total electric field vanishes vanishes at the conductor itself, but the electric field needed to drive the current to reflect the wave also vanishes! ■

8.5 Problems

1. Compute the wavelengths λ of the following sound waves using Eq. 8.11, assuming they travel through air at a speed $v = 340\text{m/s} = 0.34\text{km/s}$.

   (a) The lowest audible pitch, $\nu \approx 15\text{Hz}$.

   (b) The highest audible pitch, $\nu \approx 20\text{kHz}$.

   (c) A flute playing middle C, $\nu = 440\text{Hz}$.

2. Compute the wavelengths λ of the following electromagnetic waves.

   (a) AM radio, $\nu \approx 1000\text{kHz}$.

   (b) FM radio, $\nu \approx 100\text{MHz}$.

   (c) Microwave oven, $\nu \approx 30\text{GHz}$.

Chapter 9

Ray Optics

Ray optics or **geometric optics** is the study of light in situations where its wavelength is negligible.

9.1 Reflection

Via induced radiation, the component of light perpendicular or **normal** to a surface generates some reflection, as in Section 8.4.2, while the component parallel to the surface generates none. A surface thereby reflects light from an angle θ on one side of the normal to an angle

$$\theta' = \theta \tag{9.1}$$

on the other side of the normal, like a ball bouncing elastically from a frictionless wall, as in Fig. 9.1. In this way the surface correctly solves Maxwell's equations.

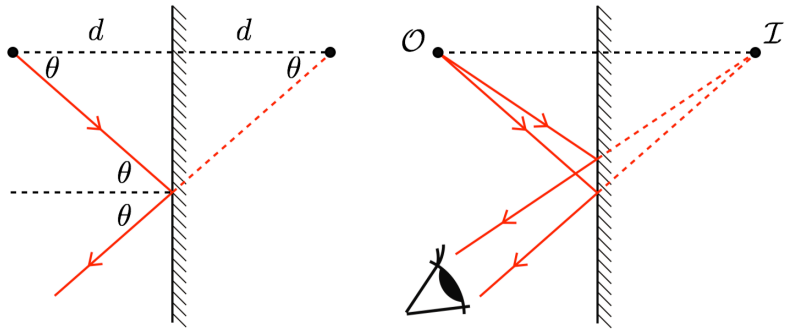


Figure 9.1: Equal-angle reflection means rays from object $\mathcal{O}$ appear to diverge from image point $\mathcal{I}$.

9.1.1 Example: Mirror Length

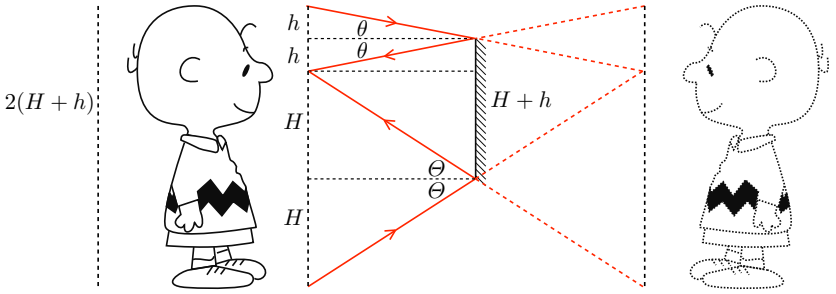


Figure 9.2: A "full length mirror" need only be half-height. (Apologies to Charles Schultz.)

Equal-angle reflection means that you need never pay for a "full length" mirror taller than half your height, as the geometry of Fig. 9.2 makes clear. ■

9.1.2 Example: Mirrors Reverse Front and Back

It is sometimes said that mirrors reverse left and right. If so, why don't they also reverse bottom and top?

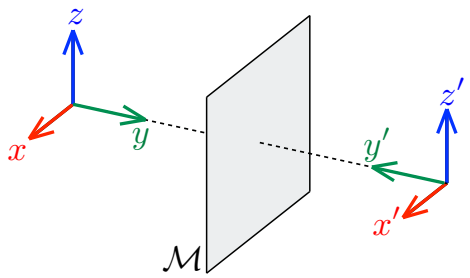


Figure 9.3: Mirrors reverse back and front, converting a right-handed coordinate system into a left-handed coordinate system: $\hat{x} \times \hat{y} = \hat{z}$ but $\hat{x}' \times \hat{y}' = -\hat{z}'$.

While it's true that the mirror image of a left-hand-glove would fit a right hand, mirrors actually reverse front and back, as in Fig. 9.3. Because we are approximately bilaterally symmetric and live in a gravitational field, when comparing image and object we tend to mentally rotate one into the the other about a vertical axis rather than a horizontal axis, thereby biasing our judgement about what mirrors do. ■

9.1.3 Example: Seeing Yourself as Others Do

In a plane mirror, you are used to seeing yourself reversed front-to-back, so if you part your hair on the left, your mirror image parts its hair on the *right*. However, consider two perpendicular mirrors, as in Fig. 9.4.

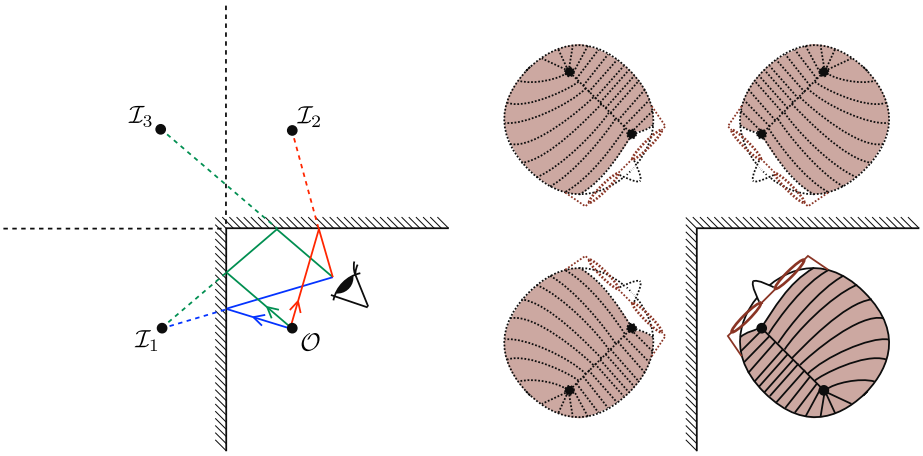


Figure 9.4: Plane mirrors reverse back and front, converting a right-handed coordinate system into a left-handed coordinate system.

The corner mirrors create three images of a point source: direct reflections in each of the two mirrors *and* a reflection of a reflection. They also create three images of an extended source, two reversed images and one non-reversed image. The latter is you as others see you. If you close your left eye, the non-reversed image also closes its left eye. ■

9.2 Refraction

From Section 8.4, light traveling in a medium of refractive index n has an effective speed $v = c/n$ and an effective wavelength

$$\lambda = \frac{v}{\nu} = \frac{c/n}{\nu} = \frac{c/\nu}{n} = \frac{\lambda}{n}. \tag{9.2}$$

Consider light incident on the boundary *between* two media. If θ and θ' are the angles the in and out light rays makes with the normal to the interface, the Fig. 9.5 triangles of hypotenuse length h imply

$$\frac{\sin\theta}{\sin\theta'} = \frac{\lambda/h}{\lambda'/h} = \frac{\lambda}{\lambda'} = \frac{\lambda/n}{\lambda/n'} = \frac{n'}{n} \tag{9.3}$$

or

$$n \sin\theta = n' \sin\theta', \tag{9.4}$$

which is **Snell's law** of refraction. For large angles, the relationship between the angles is nonlinear,

$$\theta' = \arcsin\left[\frac{n}{n'}\sin\theta\right], \tag{9.5}$$

as in the plot in Fig. 9.5. For small angles, where $\sin\theta \approx \theta \ll 1$, the angles are inversely proportional to the refractive indices,

$$\theta' \approx \frac{n}{n'}\theta \ll 1. \tag{9.6}$$

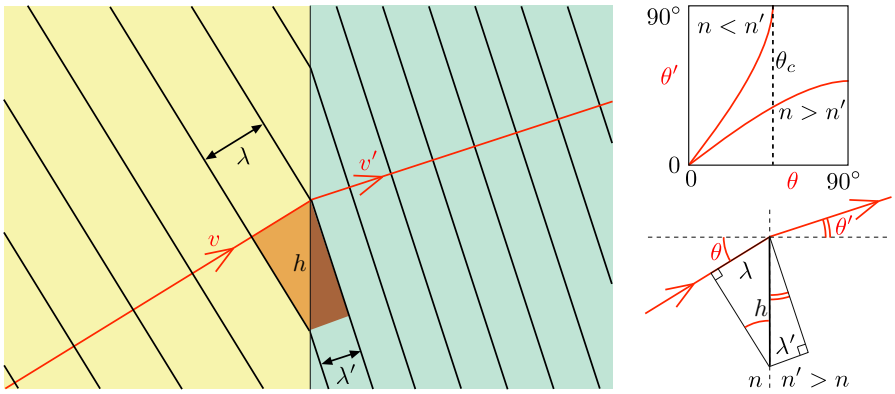


Figure 9.5: A light ray, perpendicular to its wavefronts, refracts or bends as its effective speed changes in passing from one medium to the next, with the wavefronts like ranks of a marching band walking from pavement to tall grass.

Light moving from lower to higher refractive index bends toward the normal, but light moving from higher to lower refractive index bends away from the normal, as in Fig. 9.6. In the latter case, since the maximum refracted angle is 90°, there is a critical angle θ_c such that

$$n \sin\theta_c = n' \sin 90^\circ = n' < n \tag{9.7}$$

or

$$\theta_c = \arcsin\left[\frac{n'}{n}\right] \tag{9.8}$$

beyond which refraction is frustrated and all light is perfectly reflected. This is the angle of **total internal reflection**.

In this way, the surface of a lake or pool totally reflects light from its bottom except for a small circular area directly overhead. A large refractive index implies a small angle for total internal reflection, which traps light and adds to the luster of materials like diamond.

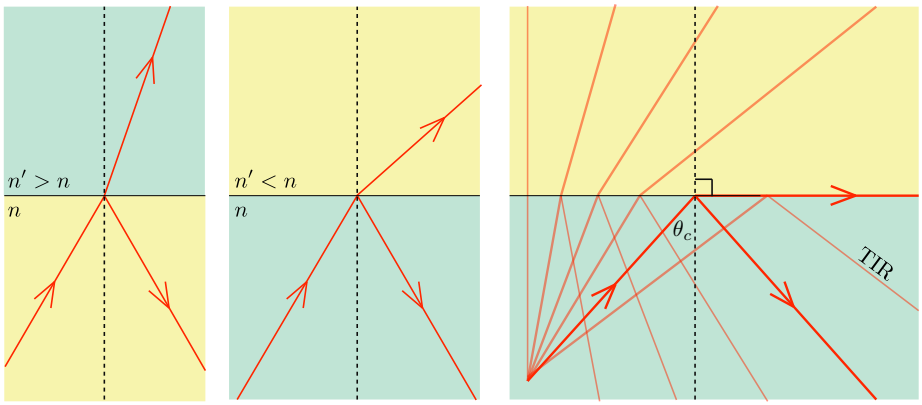


Figure 9.6: Refraction, reflection, and total internal reflection at the boundary between media of higher (blue) and lower (yellow) refraction indices.

9.2.1 Example: Apparent Depth

Consider a coin at a depth D beneath water, as in Fig. 9.7. Because light bends upon entering the water, the coin appears at an apparent depth $d < D$, closer than it really is.

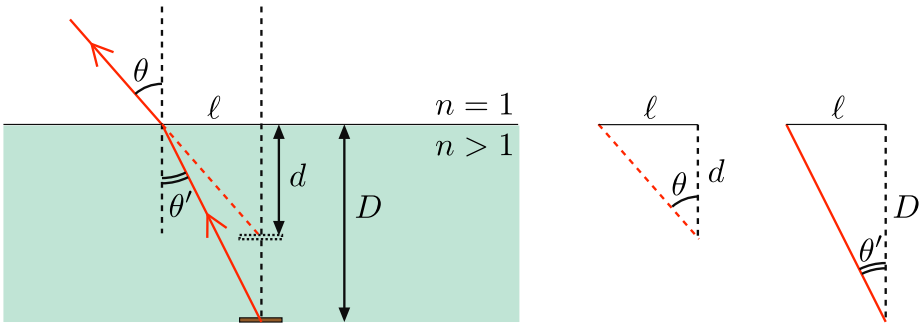


Figure 9.7: Because of refraction, the apparent depth d of a coin in water is less than the actual depth D.

Round the refractive index of air $n_a \approx 1.0003$ to unity. If the rays are near the normal, the angles are small, and Snell's law implies

$$\theta \approx 1 \sin\theta = n \sin\theta' \approx n\theta'. \tag{9.9}$$

From the geometry,

$$\theta \approx \tan\theta = \frac{\ell}{d} \tag{9.10}$$

and

$$\theta' \approx \tan\theta' = \frac{\ell}{D}. \tag{9.11}$$

Hence

$$\frac{\ell}{d} \approx n\frac{\ell}{D} \tag{9.12}$$

or

$$d \approx \frac{D}{n}. \tag{9.13}$$

For water, the coin appears to be at $1/n \approx 3/4$ its actual depth. ■

9.3 Dispersion

Refractive indices vary somewhat with color because the induced oscillation amplitude of electric charges in dielectrics vary slightly with frequency, as in the simple model of Eq. 8.40. Thus, a glass prism refracts blue light slightly more than red light and **disperses** sunlight into a **spectrum** of colors, as in Fig. 9.8. A second, inverted prism can recombine the colors into a subjective experience referred to as "white". While the intensity of the solar spectrum does peak in green light, the red, green, and blue channels of human vision in sunlight are all saturated and confer the unique "white" sensation.

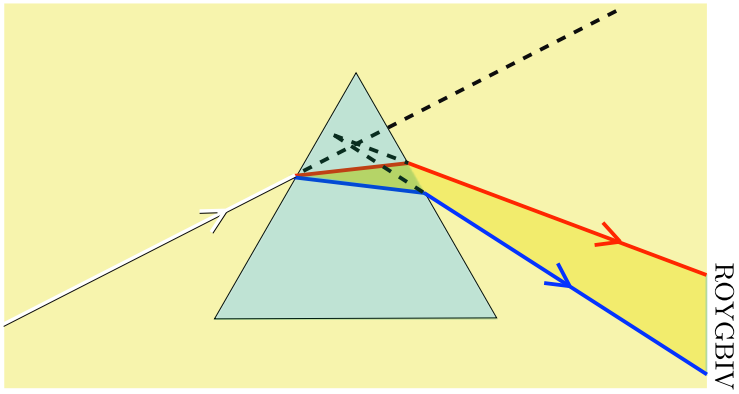


Figure 9.8: A prism disperses "white" light into its component colors by refracting shorter wavelengths more than longer wavelengths.

Perhaps the most spectacular natural examples of dispersion are **rainbows**, which appear when raindrops disperse sunlight. Primary rainbows form when sunlight refracts, reflects, and then refracts from raindrops suspended in the

atmosphere, as in Fig. 9.9. Sunlight's red component is thereby redirected or scattered through about 42°. The apparent width of the primary rainbow is about 2°, or four times the apparent diameter of a full moon. Secondary rainbows form when sunlight reflects twice from raindrops, and the extra reflection reverses the order of the colors. The gap between the primary and secondary rainbows is darker than the sky to compensate for the raindrops channelling light into the rainbows. Observers perceive rainbows as circular arcs about 42° from the shadow of the their heads. Pilots experience the full rainbow circles uninterrupted by the ground.

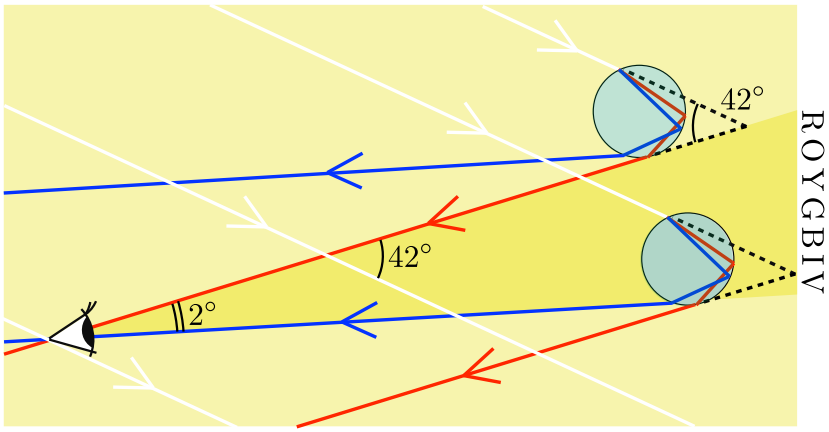


Figure 9.9: "White" sunlight refracts twice and reflects once from raindrops to form a primary rainbow. (The angles are distorted to fit the page).

9.4 Fermat's Principle

Fermat first realized that reflected and refracted rays are paths of least time, and he derived the laws of reflection and refraction as consequences. In addition, because the shortest paths between places in space are straight lines, straight line propagation also implies paths of least time.

Consider the ray rebounding from the Fig. 9.10 interface. If the speed $v = c/n$, then the rebound time

$$t = \frac{\sqrt{h^2 + (r+y)^2}}{v} + \frac{\sqrt{h^2 + (r-y)^2}}{v}, \tag{9.14}$$

where y locates an arbitrary point along the interface. Seek a minimum time by demanding

$$0 = \frac{dt}{dy} = \frac{1}{v}\frac{r+y}{\sqrt{h^2 + (r+y)^2}} - \frac{1}{v}\frac{r-y}{\sqrt{h^2 + (r-y)^2}} \tag{9.15}$$

or

$$\frac{r+y}{\sqrt{h^2+(r+y)^2}} = \frac{r-y}{\sqrt{h^2+(r-y)^2}}. \tag{9.16}$$

Square and cross multiply to get

$$(r+y)^2h^2 + (r^2-y^2) = h^2(r-y)^2 + (r^2-y^2), \tag{9.17}$$

which implies

$$(r+y)^2 = (r-y)^2 \tag{9.18}$$

and

$$y = 0 \tag{9.19}$$

indicating a midpoint reflection. Alternately, Eq. 9.16 and the Fig. 9.10 geometry imply

$$\sin\theta = \sin\theta' \tag{9.20}$$

and, as the angles are acute,

$$\theta = \theta'. \tag{9.21}$$

This is the law of reflection.

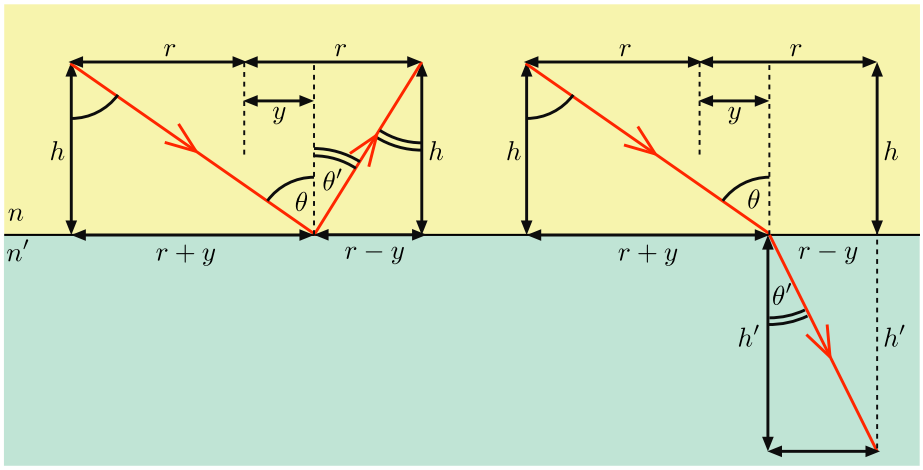


Figure 9.10: Actual paths take the least time for both reflection and refraction.

Next consider the ray penetrating the Fig. 9.10 interface. If the speed $v' = c/n'$, then the penetrating time

$$t = \frac{\sqrt{h^2+(r+y)^2}}{v} + \frac{\sqrt{h'^2+(r-y)^2}}{v'}, \tag{9.22}$$

where again y locates an arbitrary point along the interface. Seek a minimum time by demanding

$$0 = \frac{dt}{dy} = \frac{1}{v}\frac{r+y}{\sqrt{h^2+(r+y)^2}} - \frac{1}{v'}\frac{r-y}{\sqrt{h'^2+(r-y)^2}} \tag{9.23}$$

or

$$\frac{1}{v}\frac{r+y}{\sqrt{h^2+(r+y)^2}} = \frac{1}{v'}\frac{r-y}{\sqrt{h'^2+(r-y)^2}}. \tag{9.24}$$

The Fig. 9.10 geometry implies

$$\frac{\sin\theta}{v} = \frac{\sin\theta'}{v'} \tag{9.25}$$

and

$$\frac{\sin\theta}{c/n} = \frac{\sin\theta'}{c/n'} \tag{9.26}$$

and

$$n\sin\theta = n'\sin\theta'. \tag{9.27}$$

This is the law of refraction. A refracted ray travels a greater distance in the fast medium to exploit its higher speed at the expense of the longer distance.

More generally, light follows paths of *stationary* time: for reflection and refraction from plane interfaces, the times are minimal; for gravitational lensing, they are maximal; and for reflection from elliptical mirrors, they are inflectional. Fermat's principle is profoundly related to the principle of stationary **action** in mechanics and quantum mechanics.

9.4.1 Example: Mirage

Fermat's principle also applies to situations where the refractive index varies continuously, such as a gas with a temperature gradient.

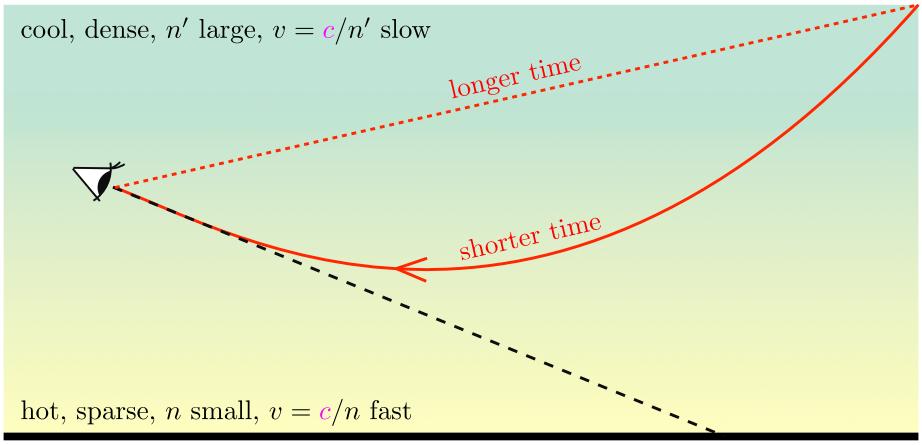


Figure 9.11: Curved least-time ray causes eye to extrapolate blue sky onto black asphalt, naturally inferring water. (Curvature greatly exaggerated.)

For example, cold air is denser than hot air, has a greater refractive index, and a slower effective light speed. Light rays will naturally avoid cold air, if possible, as in Fig. 9.11, where an asphalt road heats the air just above it providing a faster route. In a classic **mirage**, an observer looks toward the road but sees the sky, and naturally interprets the blue color as water. Strong mirages require temperature gradients of about 5°C / m. ■

9.5 Problems

1. You run toward a mirror at speed v.
   (a) With what speed does your image move relative to the mirror?
   (b) With what speed does your image move relative to *you*?

2. Light successively reflects from two perpendicular mirrors in the plane of their normals. Show that the initial and final rays are always anti-parallel. (This is the principle behind the retroreflectors Apollo astronauts placed on Luna to monitor its distance from Earth.)

3. By what fraction does the brightness of a point source increase when you place a mirror at the same distance behind it as you are in front of it? (Hint: For familiar geometric reasons, the brightness of a point source decreases like the inverse square of the distance.)

4. If a mirror rotates through an angle θ, through what angle does its reflected light rotate?

5. A light ray enters a smooth glass pane of thickness d and refractive index n at a small angle $\theta \ll 1$ relative to the normal.
   (a) Show that the exit and entrance rays are parallel.
   (b) What is their perpendicular displacement? (Hint: Use the small angle approximation for sine to simplify your work.)

6. How close horizontally can a shark at a depth h come to a seagull sitting on the surface of a smooth sea of refractive index n without the seagull seeing it?

7. A speck of dust is embedded at the center of a glass cube of refractive index n. What *minimum* fraction of the cube's surface area must you cover to hide the dust?

Chapter 10

Wave Optics

Wave optics is the study of light in situations where the wavelength is *not* negligible, including phenomena like interference and diffraction.

10.1 Babinet's Principle

Light spreads or diffracts around objects small compared to its wavelength like water waves around boulders at a beach. Consider the Fig. 10.1 opaque barrier or mask with a a hole or slit.

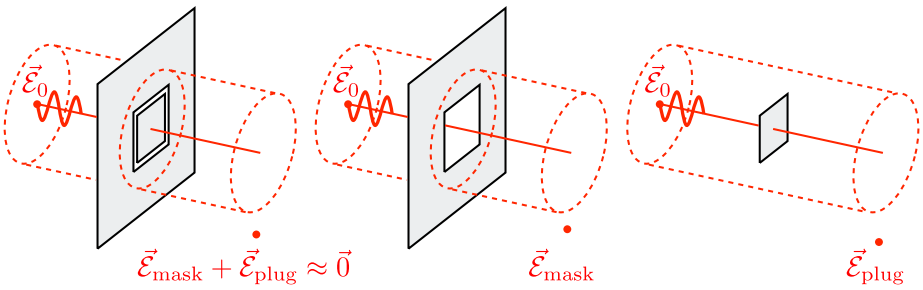


Figure 10.1: Under the excitation of an incident electromagnetic wave, complementary masks and plugs have negative electric fields and identical intensities.

Incoming electromagnetic radiation excites electrons in the barrier inducing a secondary wave that interferes with the incident wave to create a complicated **diffraction** pattern characterized by light and dark fringes. Because plugging the slit would darken the far side, the excited plug's electromagnetic field must be the negative of the barrier's field,

$$\vec{\mathcal{E}}_{\text{mask}} + \vec{\mathcal{E}}_{\text{plug}} \approx \vec{0} \tag{10.1}$$

or

$$\vec{\mathcal{E}}_{\text{mask}} \approx -\vec{\mathcal{E}}_{\text{plug}}, \tag{10.2}$$

away from the central beam. (The mask without the hole is not *exactly* equivalent to the mask plus plug as electrons can't cross the edges between them; however, this difference is negligible at large distances.)

The eye can't follow the rapid oscillations of the electric field, but is sensitive instead to the intensity $\mathcal{I}$ or average energy per unity are delivered by the incident wave. Because intensity is proportional to the *square* of the field, as in Eq. 1.20, the intensities of the mask and the plug must be the same

$$\mathcal{I}_{\text{mask}} \approx \mathcal{I}_{\text{plug}}, \tag{10.3}$$

a result known as **Babinet's principle**.

10.2 Double Slit Interference

Consider sinusoidal electromagnetic radiation of wavelength λ incident on an opaque barrier with two narrow slits of width $w \ll \lambda$ and separation $\lambda \sim d$, as in Fig. 10.2.

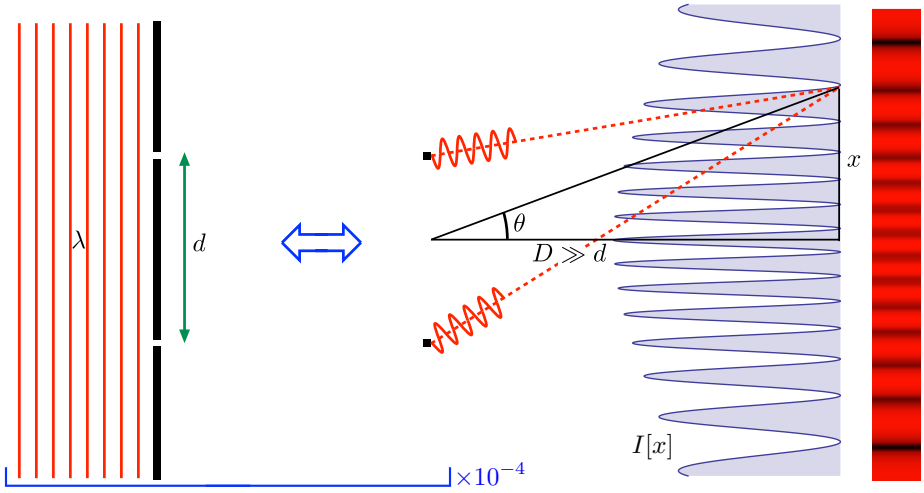


Figure 10.2: The interference pattern of the doubly slitted barrier excited by an incident wave (left) is the same as that of the coherently excited plugs (right). Interference pattern spatial extent minified 10^4 times to fit diagram.

Babinet's principle implies that the interference pattern of the doubly slitted barrier excited by an incident wave is the same as the interference pattern of the complementary and coherently excited plugs. Coherent electromagnetic waves

from the two plugs interfere constructively or destructively at a distance $D \gg d$ depending on the path difference $d \sin\theta$, as in Fig. 10.3. If adjacent paths differ by a full wavelength or several full wavelengths, constructive interference produces intensity maxima such that

$$d \sin\theta_{\max} = 0,\ \lambda, 2\lambda,\ 3\lambda,\ \cdots . \tag{10.4}$$

If adjacent paths differ by a half wavelength or half wavelength plus several full wavelengths, destructive interference produces intensity minima such that

$$d \sin\theta_{\min} = \frac{1}{2}\lambda,\ \frac{3}{2}\lambda,\ \frac{5}{2}\lambda,\ \cdots . \tag{10.5}$$

In each case,

$$1 \gg \theta \sim \tan\theta = \frac{x}{D} \tag{10.6}$$

relates the transverse position x and the angle θ.

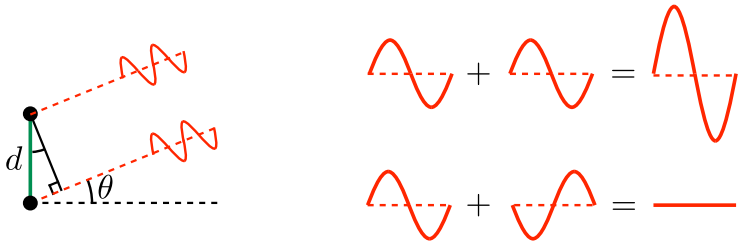


Figure 10.3: Radiation from coherent point sources interferes constructively and destructively at a distant point depending on the path difference $d \sin\theta$.

10.3 Single Slit Diffraction

Consider sinusoidal electromagnetic radiation of wavelength λ incident on an opaque barrier with a slit of width $w \sim \lambda$, as in Fig. 10.4. Babinet's principle implies that the interference pattern of the singly slit barrier excited by an incident wave is the same as the interference pattern of the complementary excited plug.

Electromagnetic waves from different portions of the plug interfere constructively or destructively at a distance $D \gg w$ to create intensity maxima or minima. Divide the plug into two halves, as in Fig. 10.5, and all waves exactly cancel in pairs if

$$\frac{w}{2} \sin\theta_{1,\min} = \frac{\lambda}{2}. \tag{10.7}$$

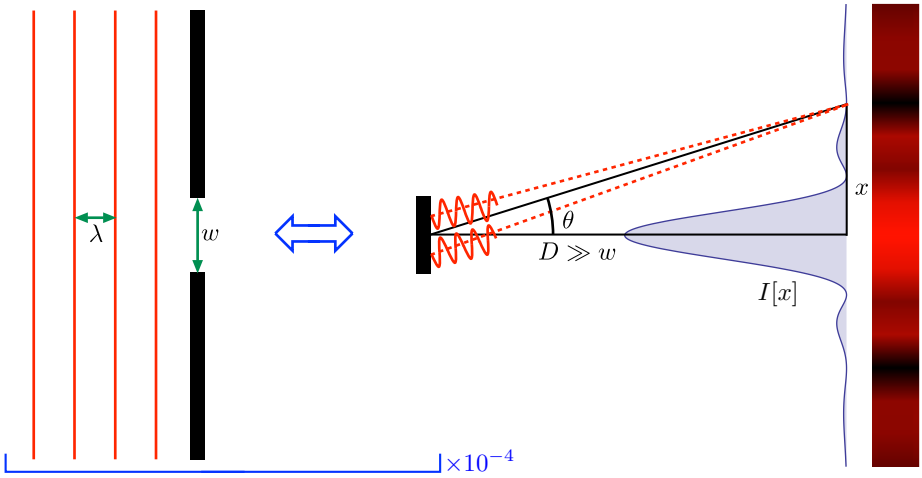


Figure 10.4: The interference pattern of the slitted barrier excited by an incident wave (left) is the same as that of the complementary excited plug (right).

Divide the plug into four quarters, as in Fig. 10.5, and all waves exactly cancel in quartets if

$$\frac{w}{4}\sin\theta_{2,\min} = \frac{\lambda}{2}. \tag{10.8}$$

More generally, single slit intensity minima satisfy

$$w\sin\theta_{\min} = \lambda,\ 2\lambda,\ 3\lambda,\ \cdots, \tag{10.9}$$

and the half-width of the central maximum is

$$1 \gg \theta_{\text{width}} \sim \frac{\lambda}{w}. \tag{10.10}$$

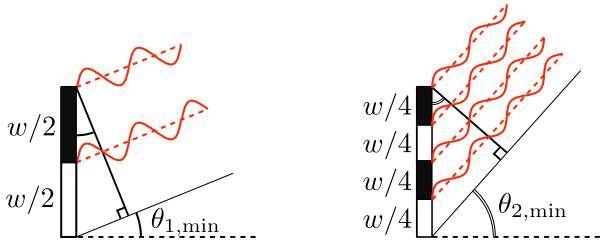


Figure 10.5: In each case, waves from the black sections of the plug cancel the corresponding waves from the white section producing an intensity minimum.

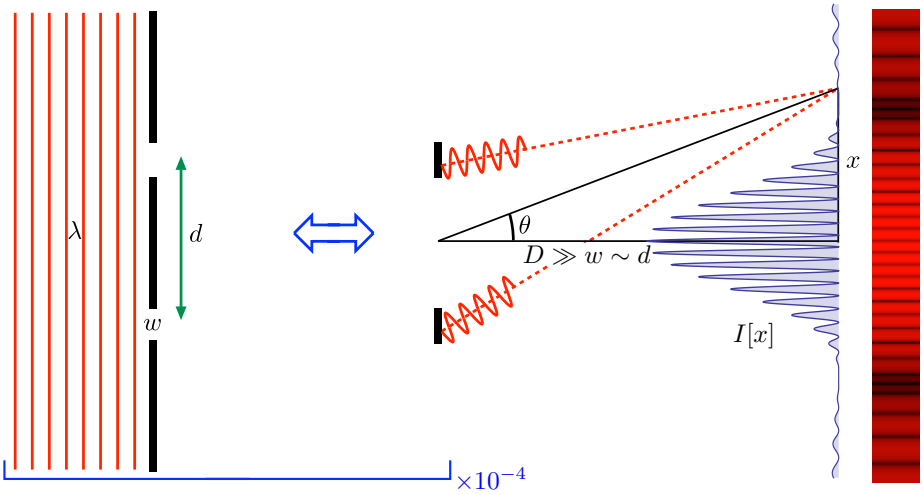


Figure 10.6: Double slit interference and single slit diffraction combined.

These single-slit diffraction minima augment the double-slit interference minima if both slit width w and separation d are comparable to the light wavelength λ. The total intensity is then the product of the intensity patter of the interference of two coherent point sources and the intensity pattern of the diffraction of an individual slit, as in Fig. 10.6. Both single slit diffraction and double slit interference are interference phenomena, but diffraction usually refers to single obstacles or holes while interference usually refers to multiple holes or sources.

Diffraction allows electromagnetic radiation to spread around corners illuminating geometric shadows with characteristic bright and dark fringes. In this way, the Voyager 1 occultation experiment dramatically demonstrated the sharpness of Saturn's rings, at centimeter wavelengths, as the spacecraft and its radio beams passed behind them as viewed from Earth.

10.4 Diffraction Grating

Consider sinusoidal electromagnetic radiation of wavelength λ incident on an opaque barrier with an array of slits of width w and separation d, as in Fig. 10.7. Babinet's principle implies that the interference pattern of the grating excited by the incident electromagnetic wave is the same as the interference pattern of an array of coherently excited plugs. Electromagnetic waves from a coherent array of points interfere constructively at a distance $D \gg w \gg d$ to create intensity maxima if the neighboring path differences

$$d \sin\theta_{\max} = 0,\ \lambda,\ 2\lambda,\ 3\lambda,\ \cdots . \tag{10.11}$$

The diffraction grating intensity maxima are at the same locations as the double slit maxima, but they are narrower and taller.

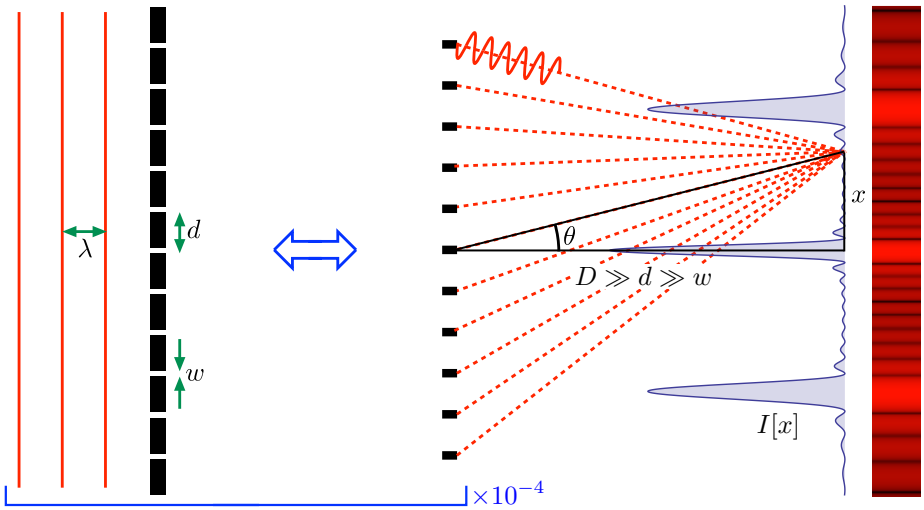


Figure 10.7: The interference pattern of a grating excited by an incident wave (left) is the same as that of an array of coherently excited plugs (right).

Like prisms, diffraction gratings can spread light into its component colors for analysis. While prisms bend blue light more than red light, diffraction gratings spread red light more than blue light, as Eq. 10.11 implies $1 \gg \theta_{1,\max} \sim \lambda/d$.

10.5 Thin Films

Beautiful and practically important effects arise when light reflects from multiple surfaces and interferes with itself, including the colors of soap bubbles and oil slicks, anti-reflection coatings on camera lenses, and the bluish iridescence of Morpho butterfly wings. A key to understanding these effects is the way light reflects from interfaces. Solving Maxwell's equations reveals that light waves reflect from a slower medium of larger index inverted, like a rope pulse from a fixed end, as in Fig. 10.8. In contrast, light waves reflect from a faster medium of smaller index non-inverted, like a rope pulse from a free end.

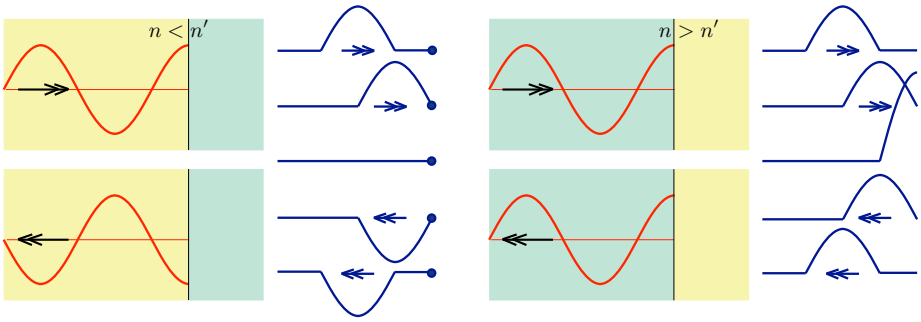


Figure 10.8: Light waves reflect from a slower medium of larger index inverted (left), like a rope pulse from a fixed end. Light waves reflect from a faster medium of smaller index non-inverted (right), like a rope pulse from a free end.

10.5.1 Example: Soap Bubble

Consider light incident on a soap bubble, as in Fig. 10.9. **Monochromatic** light of wavelength λ reflects from the outside and inside surfaces and interferes constructively and destructively to create bright and dark spots depending on the soap bubble thickness t, which typically varies due to gravity from thin at top to thick at bottom. Multi-wavelength or "white" light creates colored fringes as the different component colors are bright or dark at different places. At large thicknesses, the interference maxima and minima at even slightly different angles overlap and average out.

Light wave trains reflected from the outside and inside of the bubble are relatively shifted by the extra path length $2t$ of the inner reflection *and* the inversion at the outer, fast-slow surface but not at the inner, slow-fast surface. Since the extra inversion is equivalent to a half-wavelength shift, destructive interference and intensity minima occur when the extra path length is an integer number of wavelengths,

$$2t_{\text{min}} = 0,\ \lambda',\ 2\lambda',\ 3\lambda', \cdots \tag{10.12}$$

or

$$t_{\min} = 0,\ \frac{\lambda}{2n'},\ \frac{2\lambda}{2n'},\ \frac{3\lambda}{2n'}, \cdots , \tag{10.13}$$

where $\lambda' = \lambda/n'$ is the effective wavelength inside the soap bubble film. Constructive interference and intensity maxima occur when the extra path length is an integer number of half-wavelengths,

$$2t_{\max} = \frac{1}{2}\lambda',\ \frac{3}{2}\lambda',\ \frac{5}{2}\lambda',\ \cdots \tag{10.14}$$

or

$$t_{\max} = \frac{\lambda}{4n'},\ \frac{3\lambda}{4n'},\ \frac{5\lambda}{4n'},\ \cdots . \tag{10.15}$$

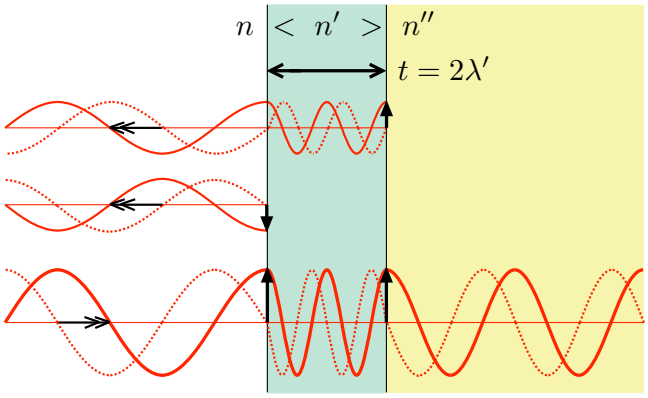


Figure 10.9: Reflections inverted from the first, fast-slow interface and not inverted from the second, slow-fast interface cancel at this wavelength.

Alternately, by rearranging Eq. 10.15, a soap bubble of thickness t is bright at the wavelengths

$$\lambda_{\max} = 4n't,\ \frac{4n't}{3},\ \frac{4n't}{5},\ \cdots . \tag{10.16}$$

For example, if $t = 340\text{nm}$ and $n' = 1.33$, then

$$\lambda_{\max} = 1810\text{nm},\ 603\text{nm},\ 362\text{nm},\ \cdots . \tag{10.17}$$

which are in the infrared, visible, and ultraviolet, so this patch of soap bubble appears green. ■

10.5.2 Example: Anti-Reflective Coating

Modern camera and telescope lenses often have many components. Unwanted reflections from such components can significantly degrade the brightnesses of the corresponding images. However, thin film coatings can cancel reflections at

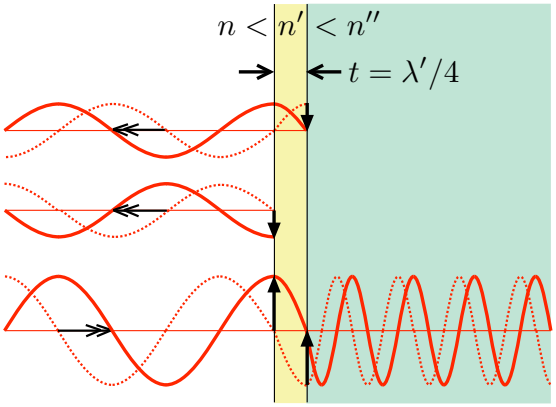


Figure 10.10: Reflections inverted from the two fast-slow interfaces cancel at this wavelength.

a single wavelength and significantly reduce them in a range of nearby wavelengths.

Consider light of wavelength λ incident on a glass lens with a magnesium fluoride MgF_2 coating of thickness t, as in Fig. 10.10. Light wave trains reflected from the coating and the lens are relatively shifted by the extra path length $2t$ of the lens reflection. Since both reflections are inverted at fast-slow surfaces, there is no extra shift. Therefore, destructive interference and intensity minima occur when the extra path length is an integer number of half-wavelengths,

$$2t_{\min} = \frac{1}{2}\lambda',\ \frac{3}{2}\lambda',\ \frac{5}{2}\lambda',\ \cdots \tag{10.18}$$

or

$$t_{\min} = \frac{\lambda}{4n'},\ \frac{3\lambda}{4n'},\ \frac{5\lambda}{4n'},\ \cdots, \tag{10.19}$$

where $\lambda' = \lambda/n'$ is the effective wavelength inside the coating. For example, if the coating index $n' = 1.38$ and the target wavelength $\lambda = 600\text{nm}$, then the thinnest antireflective coating has the quarter-wave thickness

$$t_{\min} = \frac{\lambda'}{4} = \frac{\lambda}{4n'} = 109\text{nm}. \tag{10.20}$$

■

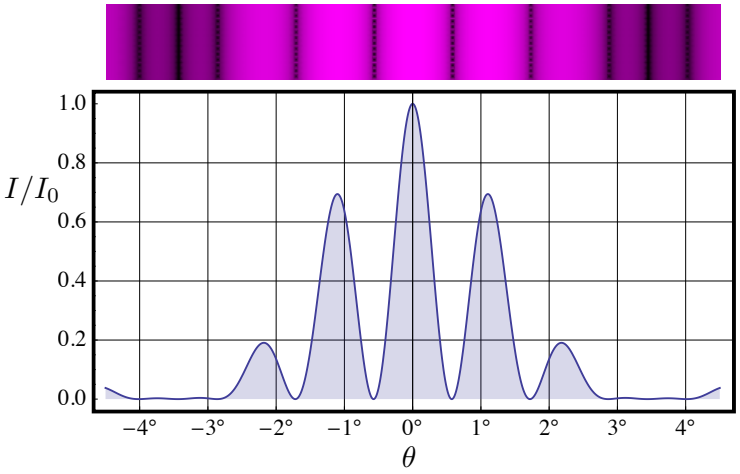
10.6 Problems

1. Violet light of wavelength 400nm illuminates a pair of slits of width w and separation d and produces the fringe pattern below. (Hint: The angles are small.)

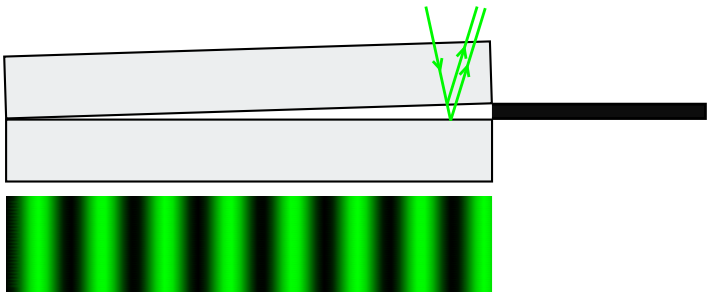
   (a) What is the ratio d/w of slit separation to width?

   (b) What is the width w of the slits?

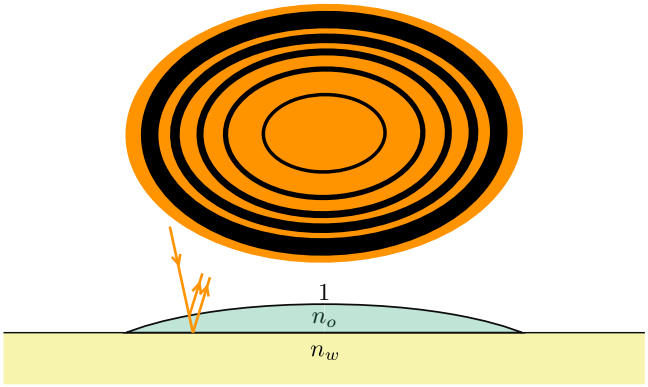
   (c) What is the distance d between the slits?



2. A paper sheet separates two optical flats at one end. A green light of wavelength $\lambda = 500$nm at normal incidence produces the bright and dark fringes shown. How thick is the paper?



3. Orange light of wavelength $\lambda = 600\text{nm}$ illuminates an iridescent gasoline slick with index $n_o = 1.25$ floating on water with index $n_w = 1.33$. How thick is the gasoline slick at the second bright fringe from the edge?

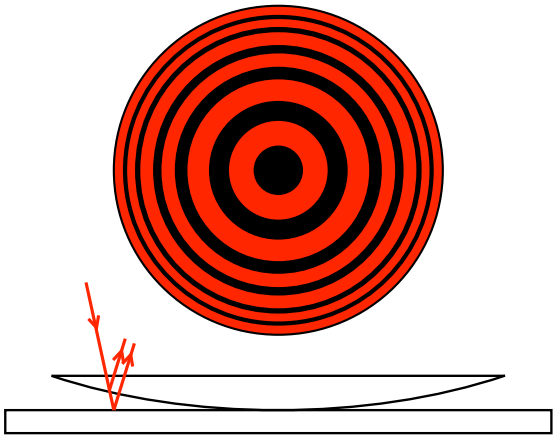


4. A convex lens rests on flat glass. Interference between light reflected from the curved lens and the flat glass interferes to form concentric bright and dark fringes called **Newton's rings**.

   (a) Why is the center dark?

   (b) Show that the radius of the mth bright ring is $r_m \approx \sqrt{(m - 1/2)\lambda R}$ for $R \gg r_m$, where λ is the light wavelength, R is the lens curvature radius, and m are natural numbers.

   (c) If red light of wavelength $\lambda = 700\text{nm}$ illuminates the lens of radius $R = 2.0\text{m}$, what is the radius of the first bright ring?



Chapter 11

Electric Energy

The "electric potential" of a charge distribution is the work per charge required to assemble it and the potential energy per charge stored in the corresponding electric field. Work, energy, and electric potential are key tools in analyzing electric circuits.

11.1 Electric Potential

Consider the electric circulation $\Gamma_{\mathcal{E}}$ of a static electric field $\vec{\mathcal{E}}$ around the closed loop joining points $\vec{a}$ and $\vec{b}$, as in Fig. 11.1.

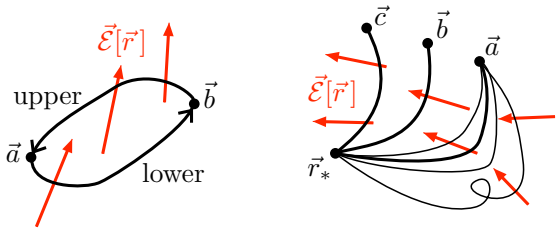


Figure 11.1: A path independent electric vector field $\vec{\mathcal{E}}$ and a reference point $\vec{r}_*$ enables the definition of the electric potential φ.

According to the Eq. 6.12 Faraday's law, in the absence of a changing magnetic field the circulation of the electric field vanishes for any closed path,

$$0 = \Gamma_{\mathcal{E}} = \oint_{\ell} \vec{\mathcal{E}} \cdot d\vec{\ell} = \underbrace{\int_{\vec{a}}^{\vec{b}} \vec{\mathcal{E}} \cdot d\vec{\ell}}_{\text{lower}} + \underbrace{\int_{\vec{b}}^{\vec{a}} \vec{\mathcal{E}} \cdot d\vec{\ell}}_{\text{upper}} \tag{11.1}$$

or

$$\underbrace{\int_{\vec{a}}^{\vec{b}} \vec{\mathcal{E}} \cdot d\vec{\ell}}_{\text{lower}} = -\underbrace{\int_{\vec{b}}^{\vec{a}} \vec{\mathcal{E}} \cdot d\vec{\ell}}_{\text{upper}} = +\underbrace{\int_{\vec{a}}^{\vec{b}} \vec{\mathcal{E}} \cdot d\vec{\ell}}_{\text{upper}}, \tag{11.2}$$

so that the line integral of the electrostatic field is the same over both the upper and lower paths. Because those paths are generic, the electrostatic field line integrals are path independent. Thus, given an electric field $\vec{\mathcal{E}}$ and a reference point $\vec{r}_*$, define a unique **electric potential** φ at point $\vec{r}$ by

$$\varphi[\vec{r}] = -\int_{\vec{r}_*}^{\vec{r}} \vec{\mathcal{E}}[\vec{r}\,'] \cdot d\vec{r}\,' = -\int_{\vec{r}_*}^{\vec{r}} \vec{\mathcal{E}} \cdot d\vec{\ell}, \tag{11.3}$$

where the minus sign is conventional (so that positive charges move from high to low potential).

Electric potential φ has SI unit of **volt**,

$$\text{unit}[\varphi] = \frac{\text{N}}{\text{C}}\text{m} = \frac{\text{J}}{\text{C}} = \text{V}, \tag{11.4}$$

which implies the electric field $\vec{\mathcal{E}}$ has SI unit of volts per meter,

$$\text{unit}[\mathcal{E}] = \frac{\text{N}}{\text{C}} = \frac{\text{V}}{\text{m}}. \tag{11.5}$$

Indeed, since the potential is proportional to the integral of the field, the field is proportional to the derivative or **gradient** of the potential.

The electric potential difference between any points $\vec{a}$ and $\vec{b}$

$$\begin{aligned}
\Delta\varphi = \varphi[\vec{b}] - \varphi[\vec{a}] &= -\int_{\vec{r}_*}^{\vec{b}} \vec{\mathcal{E}} \cdot d\vec{\ell} + \int_{\vec{r}_*}^{\vec{a}} \vec{\mathcal{E}} \cdot d\vec{\ell} \\
&= -\int_{\vec{r}_*}^{\vec{b}} \vec{\mathcal{E}} \cdot d\vec{\ell} - \int_{\vec{a}}^{\vec{r}_*} \vec{\mathcal{E}} \cdot d\vec{\ell} \\
&= -\int_{\vec{a}}^{\vec{r}_*} \vec{\mathcal{E}} \cdot d\vec{\ell} - \int_{\vec{r}_*}^{\vec{b}} \vec{\mathcal{E}} \cdot d\vec{\ell} \\
&= -\int_{\vec{a}}^{\vec{b}} \vec{\mathcal{E}} \cdot d\vec{\ell}.
\end{aligned} \tag{11.6}$$

is the negative line integral of the electric field along any path between the points. Such an electric potential difference is called a **voltage**. In simple cases when the electric field is constant, $\Delta\varphi = -\mathcal{E}_\ell \Delta\ell$ and $\mathcal{E}_\ell = -\Delta\varphi/\Delta\ell$.

Shifting the reference point $\vec{r}_*$ shifts the potential $\varphi[\vec{r}]$ without changing the electric field $\vec{\mathcal{E}}[\vec{r}]$. For example, if

$$\varphi_1[\vec{r}] = -\int_{\vec{r}_{*1}}^{\vec{r}} \vec{\mathcal{E}} \cdot d\vec{\ell}, \tag{11.7}$$

then

$$\varphi_2[\vec{r}] = -\int_{\vec{r}_{*2}}^{\vec{r}} \vec{\mathcal{E}} \cdot d\vec{\ell} = -\int_{\vec{r}_{*2}}^{\vec{r}_{*1}} \vec{\mathcal{E}} \cdot d\vec{\ell} - \int_{\vec{r}_{*1}}^{\vec{r}} \vec{\mathcal{E}} \cdot d\vec{\ell} = k + \varphi_1[\vec{r}], \tag{11.8}$$

where k is constant. Since the field is proportional to the derivative of the potential, it is unaffected by the constant shift. Although any reference point works, a convenient reference point is often at infinity (except for problems with idealized charge distributions that extend to infinity).

11.1.1 Example: Charged Shell Potential

Compute the electric potential φ of a spherical shell of charge Q and radius R centered at the origin. By symmetry, $\varphi[\vec{r}] = \varphi[r]$. Set the reference point at infinity and choose a simple radial path $d\vec{\ell} = \hat{r}dr$ to integrate over the Eq. 2.29 electric field. There are two cases. In the exterior of the shell $r > R$ and

$$\varphi[r] = -\int_{\infty}^{r>R} \mathcal{E}[r']dr' = -\int_{\infty}^{r} \epsilon_0^{-1} \frac{Q}{4\pi r'^2} dr' = +\epsilon_0^{-1} \frac{Q}{4\pi r'}\bigg|_{\infty}^{r} = \epsilon_0^{-1} \frac{Q}{4\pi r}. \quad (11.9)$$

In the interior of the shell $r < R$ and

$$\varphi[r] = -\int_{\infty}^{r<R} \mathcal{E}[r']dr' = -\int_{\infty}^{R} \epsilon_0^{-1} \frac{Q}{4\pi r'^2} dr' - \int_{R}^{r} 0\, dr' = \epsilon_0^{-1} \frac{Q}{4\pi R}. \quad (11.10)$$

Summarize this by

$$\epsilon_0 \varphi[r] = \begin{cases} \dfrac{Q}{4\pi R}, & r \le R, \\ \dfrac{Q}{4\pi r}, & r \ge R. \end{cases} \quad (11.11)$$

Unlike the electric field, which is discontinuous at the shell, the electric potential is continuous everywhere, as is shown in Fig. 11.2.

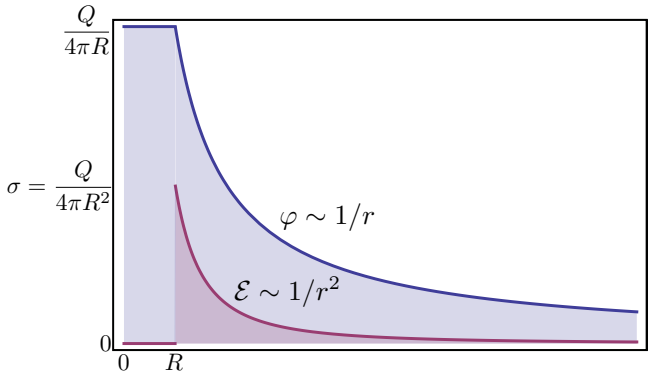


Figure 11.2: Electric field magnitude $\mathcal{E}$ and electric potential φ as a function of radial distance r for a spherical shell of radius R and charge Q.

Shrink the shell to zero, $R \to 0$, and recover the electric potential of a point charge

$$\varphi = \epsilon_0^{-1} \frac{Q}{4\pi r}, \quad (11.12)$$

as illustrated in Fig 11.3, where equipotential surfaces are orthogonal to electric field lines.

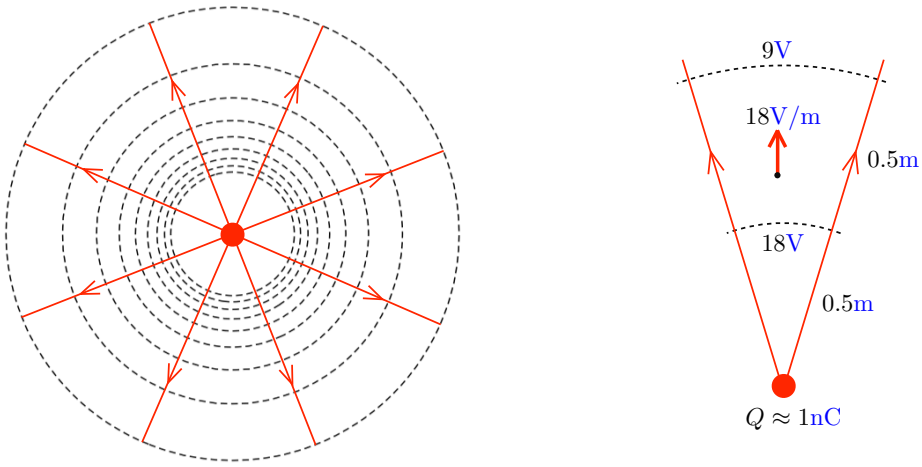


Figure 11.3: Equipotential cross sections (dashed circles) and electric field (solid radii) for a $Q \approx 1\text{nC}$ point charge. At 1m the potential is 9V. ■

11.1.2 Example: Line Charge Potential

Compute the electric potential φ of a line of charge density λ. As in Example 3.2.1, first find the electric field $\vec{\mathcal{E}}$ by applying Gauss's law $\Phi_{\mathcal{E}} = \epsilon_0^{-1} Q$ to a closed coaxial cylinder of radius $r_\perp = s$ and length ℓ to get

$$0 + \mathcal{E}(2\pi s)\ell + 0 = \epsilon_0^{-1}\ell\lambda \tag{11.13}$$

so that

$$\epsilon_0 \mathcal{E} = \frac{\lambda}{2\pi s} \tag{11.14}$$

or

$$\epsilon_0 \vec{\mathcal{E}} = \frac{\lambda}{2\pi s}\hat{s}. \tag{11.15}$$

Then integrate the electric field $\vec{\mathcal{E}}$ along a path perpendicular to the line to get

$$\epsilon_0 \varphi = -\int_{s_0}^{s} \frac{\lambda}{2\pi s'} ds' = -\frac{\lambda}{2\pi} \log s' \Big|_{s_0}^{s} = \frac{\lambda}{2\pi} \log \frac{s_0}{s}, \tag{11.16}$$

where $s_0 < \infty$ is some constant fiducial distance. ■

11.2 Potential Energy

Potential energy U stored in a charge configuration is the work W done to assemble it. Recover the stored energy as kinetic energy by letting the charges fly apart. Move a test charge q from a reference point $\vec{r}_*$, typically at infinity, at

constant velocity so that the applied force balances the electric force, $\vec{F}+q\vec{\mathcal{E}}=\vec{0}$. The work done

$$W=\int_{\vec{r}_*}^{\vec{r}}\vec{F}\cdot d\vec{\ell}=-q\int_{\vec{r}_*}^{\vec{r}}\vec{\mathcal{E}}\cdot d\vec{\ell}=q\,\varphi[\vec{r}]=U \tag{11.17}$$

is the energy stored. Thus, bringing two charges q and Q to a separation ℓ requires work

$$W=q\varphi=q\epsilon_0^{-1}\frac{Q}{4\pi\ell}=\epsilon_0^{-1}\frac{qQ}{4\pi\ell}. \tag{11.18}$$

Interpret the electric potential φ as the potential energy per unit charge stored in the charge distribution or in the corresponding electric field.

11.3 Disambiguation

Table 11.1 disambiguates three line integrals. Table 11.2 compares and contrasts similar but different definitions. The electric "potential" is the potential energy *per unit charge*; at each place in space, the vector field $\vec{\mathcal{E}}$ is the electric *force* per unit charge, and the scalar field φ is the electric *energy* per unit charge.

Table 11.1: Electric field line integrals.

integral	name
$\Gamma_{\mathcal{E}}=\oint_\ell\vec{\mathcal{E}}\cdot d\vec{\ell}$	circulation
$\varphi=-\int_{\vec{r}_*}^{\vec{r}}\vec{\mathcal{E}}\cdot d\vec{\ell}$	potential
$\Delta\varphi=-\int_{\vec{r}_1}^{\vec{r}_2}\vec{\mathcal{E}}\cdot d\vec{\ell}$	voltage

Table 11.2: Related but distinct definitions.

single charge	pair of equal charges
$\epsilon_0\varphi=\frac{Q^1}{4\pi r^1}$	$\epsilon_0 U=\frac{Q^2}{4\pi r^1}$
$\epsilon_0\mathcal{E}=\frac{Q^1}{4\pi r^2}$	$\epsilon_0 F=\frac{Q^2}{4\pi r^2}$

11.4 Problems

1. The 9MV equipotential of a 1C point charge is a sphere of what radius? Wow!

2. Compute the electric potential φ both inside and outside the charged solid sphere of Example 3.2.4.

Chapter 12

DC Circuits

In **Direct Current** or DC circuits, electric charge flows in only one direction.

12.1 Capacitance

By Eq. 11.12, a sphere of radius R having a charge Q is at the electric potential

$$\epsilon_0 \varphi = \frac{Q}{4\pi R}. \tag{12.1}$$

Equivalently, the charge stored on the sphere is

$$Q = \epsilon_0 4\pi R \varphi \tag{12.2}$$

or

$$Q = C\varphi, \tag{12.3}$$

where

$$C = \epsilon_0 4\pi R \tag{12.4}$$

is the **capacitance** of the sphere. Capacitance C has SI unit of **farad**,

$$\text{unit}[C] = \frac{\text{C}}{\text{V}} = \text{F}, \tag{12.5}$$

which is a large unit, as common capacitances are $1\mu\text{F}$ and 1nF.

Capacitance $C = Q/\varphi$ is large if an object can store a large charge Q at low potential φ. For example, the capacitance of Earth,

$$\begin{aligned} C_\oplus &= 4\pi R_\oplus \epsilon_0 \\ &= 4\pi \left(6.4 \times 10^6 \text{m}\right) \left(8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N m}^2}\right) \\ &= 7.1 \times 10^{-4} \frac{\text{C}}{\text{J/C}} \\ &= 0.71\text{mF}, \end{aligned} \tag{12.6}$$

is about a tenth of a millifarad. Thus, Earth can store an extra 1mC of charge at a potential of 1V. Alternately, it takes 1mC of charge to raise Earth's potential 1V.

12.2 Capacitors

Most common **capacitors** are insulated metallic conductors with opposite electric charge $\pm Q$, potential difference or voltage $\Delta\varphi$, and capacitance

$$C = \frac{Q}{\Delta\varphi}, \tag{12.7}$$

where conventionally $\Delta\varphi > 0$ and $Q > 0$. As with isolated objects, the capacitances of common capacitors is large if they can store large charges at small voltages. A short name for capacitors is "caps".

12.2.1 Example: Parallel Plate Capacitor

Consider a conducting square of area A and charge Q a perpendicular distance d from a similar conducting square of charge $-Q$, as in Fig. 12.1.

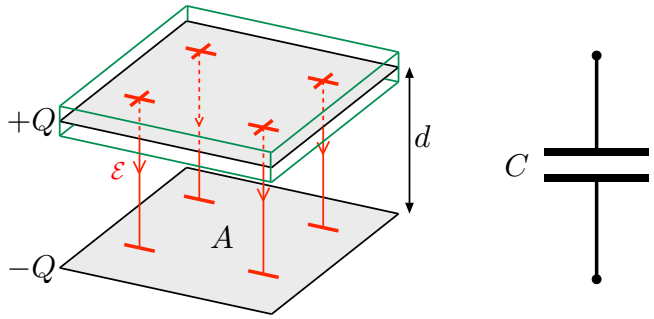


Figure 12.1: Parallel plate capacitor with Gaussian surface (left). Capacitor circuit symbol (right) is a pair of parallel lines reminiscent of parallel plates.

Apply the Eq. 3.12 Gauss's law

$$\epsilon_0 \Phi_{\mathcal{E}} = Q \tag{12.8}$$

to the cuboid straddling the top square to find

$$\epsilon_0 \mathcal{E} A = \sigma A \tag{12.9}$$

and the electric field

$$\epsilon_0 \mathcal{E} = \sigma = \frac{Q}{A}. \tag{12.10}$$

Integrate the electric field to find the voltage

$$\Delta\varphi = -\int_{\text{bot}}^{\text{top}} \vec{\mathcal{E}} \cdot d\vec{\ell} = +\mathcal{E}d = \epsilon_0^{-1}\frac{Qd}{A}. \tag{12.11}$$

Hence, the capacitance

$$C = \frac{Q}{\Delta\varphi} = \frac{Q}{\epsilon_0^{-1}Qd/A} = \epsilon_0\frac{A}{d} \tag{12.12}$$

is proportional to the squares' common area A and inversely proportional to their separation d. Large plates separated by a small distance can store a large charge at low potential. Practical parallel plate capacitors are often sandwiches of two metal conductors separated by insulating plastic and compactly rolled into cylinders. ■

Combine capacitors in simple circuits in various ways, as in Fig. 12.2. Such combinations can be replaced by single equivalent capacitors.

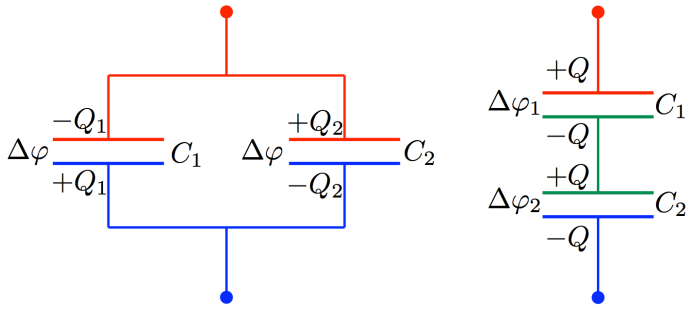


Figure 12.2: Capacitors in parallel (left) and series (right) can be replaced by single equivalent capacitors. Colors code conductor equipotentials.

For the Fig. 12.2 capacitors in **parallel**, the voltages are the same,

$$\Delta\varphi = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}, \tag{12.13}$$

and the charges add,

$$Q = Q_1 + Q_2 = C_1\Delta\varphi + C_2\Delta\varphi = (C_1 + C_2)\Delta\varphi = C_p\Delta\varphi, \tag{12.14}$$

where the equivalent capacitance

$$C_p = C_1 + C_2, \tag{12.15}$$

For N capacitors in parallel, the equivalent capacitance

$$C_p = C_1 + C_2 + C_3 + \cdots + C_N. \tag{12.16}$$

For capacitors in parallel, the equivalent capacitance is the sum of the capacitances and is always greater than largest individual capacitance,

$$C_p > \max_n C_n. \tag{12.17}$$

Adding capacitors in parallel *increases* the equivalent capacitance. For example, if $C_1 = C_2 = C$, then $C_p = 2C$.

For the Fig. 12.2 capacitors in **series**, the charges are the same,

$$Q = C_1 \Delta\varphi_1 = C_1 \Delta\varphi_2 \tag{12.18}$$

and the voltages add,

$$\Delta\varphi = \Delta\varphi_1 + \Delta\varphi_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C_s}, \tag{12.19}$$

where the equivalent inverse capacitance

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{12.20}$$

For N capacitors in series, the equivalent inverse capacitance

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \cdots + \frac{1}{C_N}. \tag{12.21}$$

For capacitors in series, the equivalent inverse capacitance is the sum of the inverse capacitances, and the equivalent capacitance is always *less* than the smallest individual capacitance,

$$C_s < \min_n C_n. \tag{12.22}$$

Adding capacitors in series *decreases* the equivalent capacitance. For example, if $C_1 = C_2 = C$, then $C_s = C/2$.

A capacitor is *not* a battery. A capacitor accumulates a fixed amount of charge, and its voltage decreases during discharge. A battery produces charge via electrochemical reactions, and it maintains a constant voltage during discharge. Capacitors store charge while batteries "pump" charge. Capacitors provide quick bursts of energy, as in a camera flash, while batteries provide steady energy for long times, as in a laptop computer.

12.3 Resistance

Consider a wire of length ℓ and cross sectional area A in an electric field $\vec{\mathcal{E}}$ due to a voltage $\Delta\varphi = \mathcal{E}\ell$, as in Fig. 12.3. The electric field accelerates free electrons between collisions with a lattice of positive nuclei. Due to friction between the gas and the lattice, this electron "gas" drifts slowly at centimeters per second even though the individual electrons move quickly at millions of kilometers per

hour. The resulting current is proportional to the voltage ΔV and area A and inversely proportional to the length ℓ,

$$I \propto \frac{A}{\ell}\Delta\varphi, \tag{12.23}$$

or

$$I = \frac{1}{R}\Delta\varphi \tag{12.24}$$

where the the extrinsic **resistance**

$$\mathcal{R} = \rho\frac{\ell}{A}, \tag{12.25}$$

and the intrinsic **resistivity** ρ depends on the material, which is referred to as **ohmic**. Resistance $\mathcal{R}$ has SI unit of **ohm**,

$$\text{unit}[\mathcal{R}] = \frac{\text{V}}{\text{A}} = \Omega. \tag{12.26}$$

Important non-ohmic devices include diodes, whose current is *not* simply proportional to the applied voltage. Ohmic resistance $\mathcal{R} = \Delta\varphi/I$ is large if small currents I generate large voltages $\Delta\varphi$.

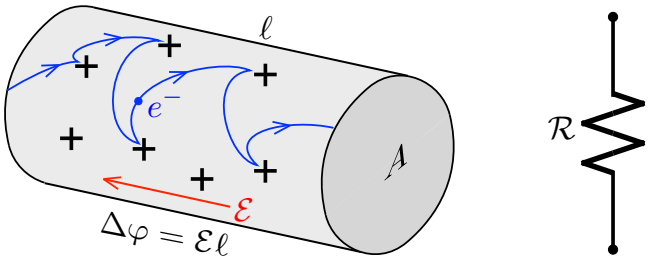


Figure 12.3: Wire or ohmic resistor and drifting electron (left). Resistor circuit symbol (right) is a zigzag reminiscent of the electron's path through the resistor.

12.4 Resistors

Metallic conductors like copper wire have some resistance, but electronic devices typically have "lumped" high resistance components called **resistors** that control and modify current. Resistors are often short pieces of **graphite**, a common allotrope of carbon.

Combine resistors in simple circuits in various ways, as in Fig. 12.4. Such combinations can be replaced by single equivalent resistors. For the Fig. 12.4 capacitors in **parallel**, the voltages are the same,

$$\Delta\varphi = I_1\mathcal{R}_1 = I_2\mathcal{R}_2, \tag{12.27}$$

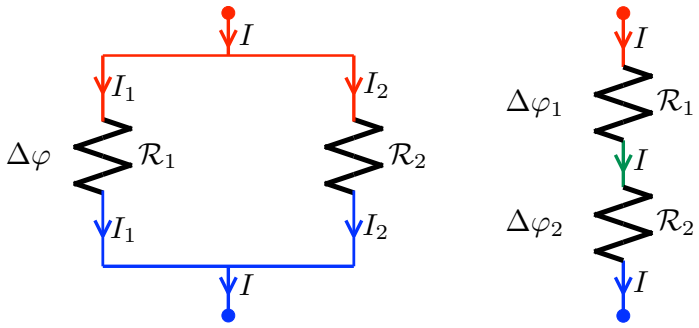


Figure 12.4: Resistors in parallel (left) and series (right) can be replaced by single equivalent resistors. Colors code conductor equipotentials.

and the currents add,

$$I = I_1 + I_2 = \frac{\Delta\varphi}{\mathcal{R}_1} + \frac{\Delta\varphi}{\mathcal{R}_2} = \left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}\right)\Delta\varphi = \mathcal{R}_p\Delta\varphi, \tag{12.28}$$

where the equivalent inverse resistance

$$\frac{1}{\mathcal{R}_p} = \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}. \tag{12.29}$$

For N resistors in parallel, the equivalent inverse resistance

$$\frac{1}{\mathcal{R}_p} = \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} + \frac{1}{\mathcal{R}_3}\cdots + \frac{1}{\mathcal{R}_N}. \tag{12.30}$$

For resistors in parallel, the equivalent inverse resistance is the sum of the inverse resistances, and the equivalent resistance is always *less* than the smallest individual resistance,

$$\mathcal{R}_p < \min_n \mathcal{R}_n. \tag{12.31}$$

Adding resistors in parallel *decreases* the equivalent resistance. For example, if $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$, then $\mathcal{R}_p = \mathcal{R}/2$.

For the Fig. 12.4 resistors in **series**, the currents are the same,

$$I = \frac{\Delta\varphi_1}{\mathcal{R}_1} = \frac{\Delta\varphi_2}{\mathcal{R}_2} \tag{12.32}$$

and the voltages add,

$$\Delta\varphi = \Delta\varphi_1 + \Delta\varphi_2 = I\mathcal{R}_1 + I\mathcal{R}_2 = I(\mathcal{R}_1 + \mathcal{R}_2) = I\mathcal{R}_s, \tag{12.33}$$

where the equivalent resistance

$$\mathcal{R}_s = \mathcal{R}_1 + \mathcal{R}_2. \tag{12.34}$$

For N resistors in series, the equivalent resistance

$$\mathcal{R}_s = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_N. \tag{12.35}$$

For resistors in series, the equivalent resistance is the sum of the resistances, and the equivalent resistance is always *greater* than the largest individual resistance,

$$\mathcal{R}_s > \max_n \mathcal{R}_n. \tag{12.36}$$

Adding resistors in series *increases* the equivalent resistance. For example, if $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$, then $\mathcal{R}_s = 2\mathcal{R}$.

A hydrodynamic analogy makes the parallel and series resistance formulas plausible: current flowing in the wires is like water flowing in pipes; flow resistance increases as pipes are added in series and decreases as pipes are added in parallel. Note that the Eq. 12.16 and Eq. 12.21 capacitance combination rules are interchanged relative to the Eq. 12.30 and Eq. 12.35 resistance combination rules.

12.4.1 Example: Infinite Resistance Ladder

Let $\mathcal{R}_e$ be the equivalent resistance of an infinite ladder of identical resistors $\mathcal{R}$, as in Fig. 12.5. Since an infinite set is equivalent to a proper subset of itself, replace the infinite ladder with an equivalent series-parallel combination whose equivalent inverse resistance satisfies

$$\frac{1}{\mathcal{R}_e} = \frac{1}{\mathcal{R}} + \frac{1}{2\mathcal{R} + \mathcal{R}_e}. \tag{12.37}$$

Clear the fractions to obtain the quadratic equation

$$\mathcal{R}_e^2 + 2\mathcal{R}\mathcal{R}_e - 2\mathcal{R}^2 = 0, \tag{12.38}$$

which has one positive root

$$\mathcal{R}_e = (\sqrt{3} - 1)\mathcal{R} \approx 0.732\mathcal{R} < \mathcal{R}. \tag{12.39}$$

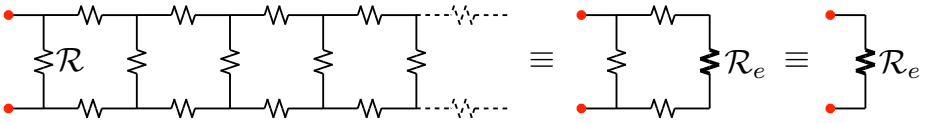


Figure 12.5: Infinite ladder of identical resistors and equivalent resistor combinations. ■

12.4.2 Example: Infinite Resistance Grid

Let $\mathcal{R}_e$ be the equivalent resistance between adjacent nodes of an infinite grid of identical resistors $\mathcal{R}$, as in Fig. 12.6. Exploit symmetry and superposition. Inject current I at node m and remove it at infinity. By symmetry, the current splits into quarters at m so that the voltage from m to n is

$$\Delta\varphi_{m\infty} = \frac{I}{4}\mathcal{R}. \tag{12.40}$$

Similarly, inject current I at infinity and remove it at node n. By symmetry, the current converges in quarters at n so that the voltage from m to n is again

$$\Delta\varphi_{\infty n} = \frac{I}{4}\mathcal{R}. \tag{12.41}$$

Superpose these two injections to get current I entering node m and exiting node n with voltage

$$\Delta\varphi_{mn} = \Delta\varphi_{m\infty} + \Delta\varphi_{\infty n} = \frac{I}{4}\mathcal{R} + \frac{I}{4}\mathcal{R} = \frac{1}{2}I\mathcal{R} \equiv I\mathcal{R}_e, \tag{12.42}$$

where the equivalent resistance

$$\mathcal{R}_e = \frac{1}{2}\mathcal{R} < (\sqrt{3}-1)\mathcal{R} < \mathcal{R}. \tag{12.43}$$

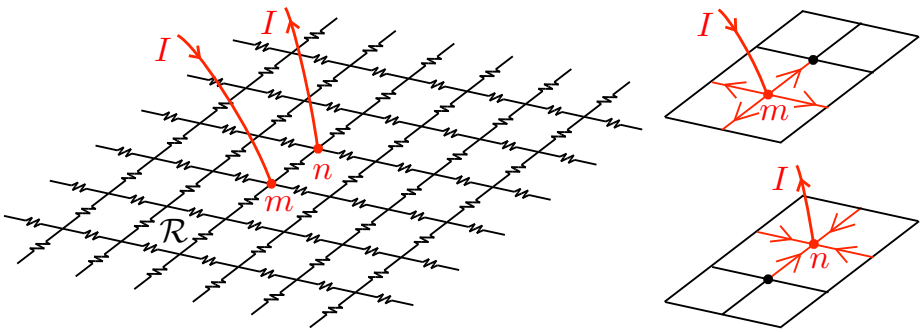


Figure 12.6: Infinite grid of identical resistors with current injected at node m and withdrawn at node n. ■

12.5 Conductance

Conductance is the inverse of resistance,

$$G = \frac{1}{\mathcal{R}}, \tag{12.44}$$

and hence is an alternative that doesn't contain new information. However, conductance is more analogous to capacitance then resistance, as in Table 12.1, and thereby simplifies learning simple circuit theory. In addition, the slopes of an electrical engineer's current versus voltage plot or "I-V curve" are conductances not resistances.

Table 12.1: Capacitance and conductance are analogous.

Capacitance	Conductance
$C = \dfrac{Q}{\Delta\varphi}$ $C \propto \dfrac{A}{d}$	$G = \dfrac{I}{\Delta\varphi}$ $G \propto \dfrac{A}{\ell}$
$\text{unit}[C] = \dfrac{\text{C}}{\text{V}} = \text{F}$ farad	$\text{unit}[G] = \dfrac{\text{A}}{\text{V}} = \text{S}$ siemens
$C_P = C_1 + C_2$ $\dfrac{1}{C_S} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$	$G_P = G_1 + G_2$ $\dfrac{1}{G_S} = \dfrac{1}{G_1} + \dfrac{1}{G_2}$

12.6 Kirchoff's Rules

Kirchoff's rules facilitate analysis of complicated circuits, even those that cannot be decomposed into parallel and series combinations. They follow from energy and charge conservation. Around any circuit **loop**, energy conservation requires that the sum of all electric potential differences or voltages must vanish,

$$\sum_{\text{loop}} \Delta\varphi_n = 0. \tag{12.45}$$

At any **node**, charge conservation requires that the algebraic sum of all the currents (positive for in and negative for out) must vanish,

$$\sum_{\text{node}} I_n = 0. \tag{12.46}$$

Alternately, at any node, the sum of all the currents in must equal the sum of all the currents out.

12.6.1 Example: Resistor Cube

Let $\mathcal{R}_{13}$ be the equivalent resistance across a face of a cube of identical resistors $\mathcal{R}$, as in Fig. 12.7.

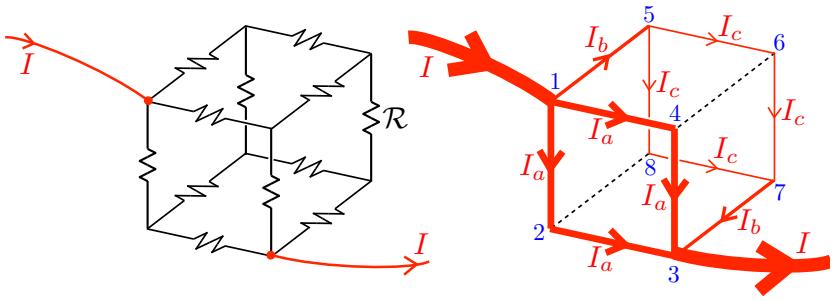


Figure 12.7: Cube of identical resistors with current injected across a face diagonal (left). Relative thicknesses of lines denote currents (right).

By symmetry, the currents satisfy

$$I_{12} = I_{14} = I_{23} = I_{43} \equiv I_a, \tag{12.47a}$$
$$I_{15} = I_{73} \equiv I_b, \tag{12.47b}$$
$$I_{56} = I_{58} = I_{87} = I_{67} \equiv I_c, \tag{12.47c}$$
$$I_{84} = I_{62} = 0, \tag{12.47d}$$

and the potentials satisfy

$$\varphi_2 = \varphi_4, \tag{12.48a}$$
$$\varphi_6 = \varphi_8. \tag{12.48b}$$

Kirchoff's loop rule implies

$$\Delta\varphi_{13} = I_a\mathcal{R} + I_a\mathcal{R} \tag{12.49}$$

and

$$\Delta\varphi_{13} = I_b\mathcal{R} + I_c\mathcal{R} + I_c\mathcal{R} + I_b\mathcal{R}. \tag{12.50}$$

Kirchoff's node rule implies

$$I = I_a + I_a + I_b \tag{12.51}$$

and

$$I_b = I_c + I_c. \tag{12.52}$$

Collect these into three equations

$$2I_a = 2I_b + 2I_c, \tag{12.53a}$$
$$I = 2I_a + I_b, \tag{12.53b}$$
$$I_b = 2I_c, \tag{12.53c}$$

and simultaneously solve for the three unknown currents

$$I_a = \frac{3}{8} I, \tag{12.54a}$$

$$I_b = \frac{1}{4} I, \tag{12.54b}$$

$$I_c = \frac{1}{8} I. \tag{12.54c}$$

Hence, the voltage across the face

$$\Delta\phi_{13} = I\mathcal{R}_{13} = 2I_a\mathcal{R} = \frac{3}{4} I\mathcal{R}, \tag{12.55}$$

and so the equivalent resistance

$$\mathcal{R}_{13} = \frac{3}{4}\mathcal{R}. \tag{12.56}$$

Obtain the same result using equivalent parallel and series resistor combinations by exploiting Eq. 12.48 to fuse the corresponding nodes. ■

12.7 Power

Consider the simple circuit of Fig.12.8. In the **battery**, chemical reactions pump charges into the circuit at constant voltage. The battery circuit symbol is a stack of long and short lines reminiscent of the early voltaic piles of dissimilar metals. (A longer line represents the positive terminal of the battery as the "+" sign requires more "line" than the "−" sign.)

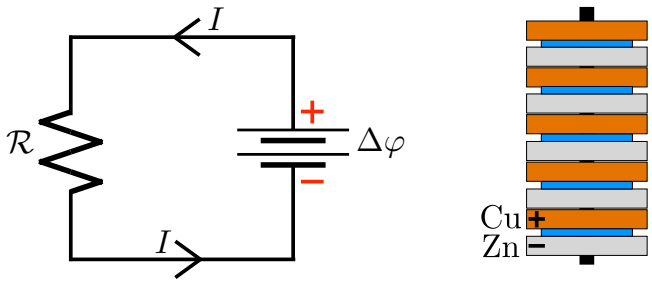


Figure 12.8: Chemical battery produces a voltage $\Delta\varphi$ that drives current I through a load $\mathcal{R}$ (left). A voltaic pile is a simple battery (right).

If a battery raises a charge dQ through a voltage $\Delta\varphi$, it does work

$$dW = dQ\Delta\varphi \tag{12.57}$$

at a rate

$$\frac{dW}{dt} = \frac{dQ}{dt}\Delta\varphi \tag{12.58}$$

to deliver power

$$P = I\Delta\varphi. \tag{12.59}$$

Inside the resistor, the charge dQ drops through a potential difference $\Delta\varphi$ losing energy

$$dQ\Delta\varphi = dU \tag{12.60}$$

at a rate

$$\frac{dQ}{dt}\Delta\varphi = \frac{dU}{dt} \tag{12.61}$$

to dissipate power

$$I\Delta\varphi = P. \tag{12.62}$$

A hydrodynamic analogy is a pump raising water to a height from which it falls back down by doing work to rotate a water wheel. For ohmic devices,

$$\Delta\varphi = I\mathcal{R} \tag{12.63}$$

and

$$P = I^2\mathcal{R} = \frac{(\Delta\varphi)^2}{\mathcal{R}}. \tag{12.64}$$

12.7.1 Example: Power Transmission

A power plant delivers power P_d at voltage $\Delta\varphi$ to a city along transmission lines with total resistance $\mathcal{R}$. The current through the city is

$$I = \frac{P_d}{\Delta\varphi}. \tag{12.65}$$

The power lost heating the wires is

$$P_\ell = I^2\mathcal{R} = \left(\frac{P_d}{\Delta\varphi}\right)^2 \mathcal{R} \propto (\Delta\varphi)^{-2}. \tag{12.66}$$

Thus high voltage transmission lines minimize power losses, which can otherwise be significant or even dominant. Even so, delivering 100MW at 100kV over 10Ω lines incurs 10% loss. ■

12.7.2 Example: Minimum Power Dissipation

How does a current I_0 divide at a circuit node that branches to a parallel combination of resistors $\mathcal{R}_1$ and $\mathcal{R}_2$? Kirchoff's node and loop rules imply

$$I_0 - I_1 - I_2 = 0 \tag{12.67}$$

and

$$I_1\mathcal{R}_1 - I_2\mathcal{R}_2 = 0. \tag{12.68}$$

which imply the currents

$$I_1 = \frac{\mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} I_0, \tag{12.69a}$$

$$I_2 = \frac{\mathcal{R}_1}{\mathcal{R}_1 + \mathcal{R}_2} I_0. \tag{12.69b}$$

Note that the branch currents are inverse to the resistances, $I_1/I_2 = \mathcal{R}_1/\mathcal{R}_2$, and $I_1 = I_2 = I_0/2$ if $\mathcal{R}_1 = \mathcal{R}_2$.

Alternately, consider the power dissipated

$$P = I_1^2 \mathcal{R}_1 + I_2^2 \mathcal{R}_2 = I_1^2 \mathcal{R}_1 + (I_0 - I_1)^2 \mathcal{R}_2. \tag{12.70}$$

Minimize this by demanding

$$0 = \frac{dP}{dI_1} = 2I_1 \mathcal{R}_1 - 2(I_0 - I_1)\mathcal{R}_2 \tag{12.71}$$

and checking

$$\frac{d^2 P}{dI_1^2} = 2\mathcal{R}_1 + 2\mathcal{R}_2 > 0, \tag{12.72}$$

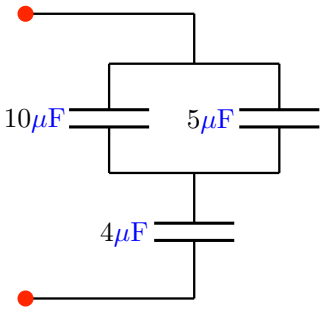
which implies

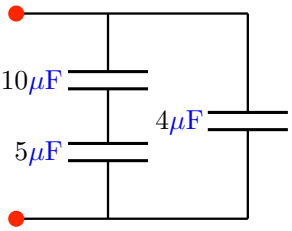
$$I_1 = \frac{\mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} I_0, \tag{12.73}$$

as before. More generally, current distributes itself in a network so as to minimize total power dissipation. ■

12.8 Problems

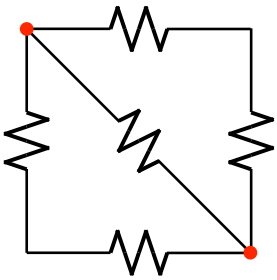
1. What is the equivalent capacitance of these series-parallel combinations of capacitors?

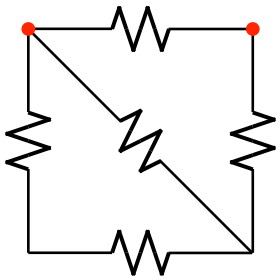




2. How many $1\mu F$ capacitors are needed to store $1C$ of charge at $110V$?

3. What is the equivalent resistance of these series-parallel combinations of 1Ω resistors across the diagonal (left) and edge (right)?





4. What is the equivalent resistance across the body diagonal of a cube of 1Ω resistors?

Chapter 13

AC Circuits

In **Alternating Current** or AC circuits, electrical current periodically reverses direction.

13.1 Generator

A wire loop rotating about about an axis perpendicular to a magnetic field intercepts a sinusoidally varying magnetic flux

$$\Phi_{\mathcal{B}} = \iint_A \vec{\mathcal{B}} \cdot d\vec{a} = \mathcal{B}A\cos\omega t. \tag{13.1}$$

By Faraday's law, the sinusoidal magnetic flux induces a sinusoidal electric circulation

$$I\mathcal{R} = \Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}} = -\frac{d\Phi_{\mathcal{B}}}{dt} = -\omega\mathcal{B}A\sin\omega t, \tag{13.2}$$

which can drive current in an AC circuit, as in Fig. 13.1.

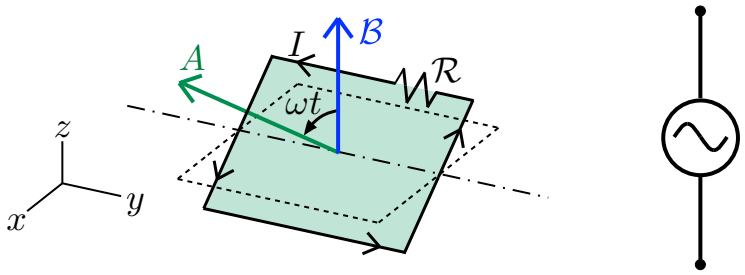


Figure 13.1: A wire loop rotating in a magnetic field drives a sinusoidal current (left). AC generator symbol (right) features one period of a sine curve.

13.2 Inductance

A coil of wire in an AC circuit is an **inductor**. If the current through the coil changes, then so will the corresponding magnetic field. The coil's magnet flux is proportional to its current, so

$$\Phi_{\mathcal{B}} = LI, \tag{13.3}$$

where the **inductance** $L > 0$ is the proportionality constant. The inductance has SI unit of **henry**,

$$\text{unit}[L] = \frac{\text{T m}^2}{\text{A}} = \text{H}. \tag{13.4}$$

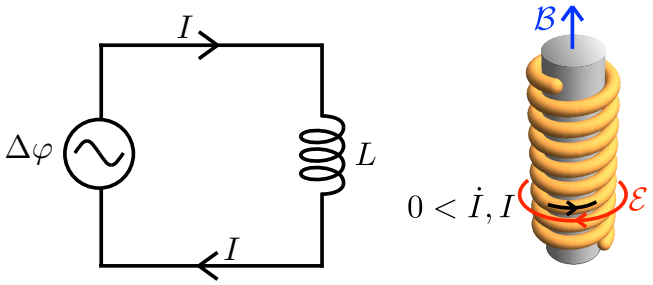


Figure 13.2: An inductor resists changes in an AC circuit (left). A wire coiled around a core is a simple inductor (right).

By Faraday's law, the changing magnetic flux induces a circulating electric field, as in Fig. 13.2, which opposes the current change. For one turn of the inductor, the counter voltage

$$\Delta\varphi_1 = \int_{\text{turn}} \vec{\mathcal{E}} \cdot d\vec{\ell} \approx \oint_{\text{circle}} \vec{\mathcal{E}} \cdot d\vec{\ell} = \Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}1} = -L_1 \dot{I} = -L_1 \frac{dI}{dt} \tag{13.5}$$

For N turns, the counter voltage

$$\Delta\varphi = -L\frac{dI}{dt}, \tag{13.6}$$

where $L = NL_1$. Inductance $L = -\Delta\varphi/\dot{I}$ is large if slowly changing currents $\dot{I}$ generate large counter voltages $\Delta\varphi$.

Isolated inductors combine like resistors in series and parallel. For isolated inductors in parallel, the voltages are the same and the currents add, so the equivalent inverse inductance is the sum of the inverse inductances

$$\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \cdots + \frac{1}{L_N}, \tag{13.7}$$

and the equivalent inductance is always *less* than the smallest individual inductance. For isolated inductors in series, the currents are the same and the voltages add, so the equivalent inductance is the sum of the inductances

$$L_s = L_1 + L_2 + L_3 + \cdots + L_N, \tag{13.8}$$

and the equivalent inductance is always *greater* than the largest individual inductance.

Table 13.1: Lumped circuit elements.

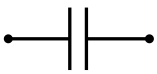
capacitor	resistor	inductor
		
$\Delta\varphi = \dfrac{Q}{C}$	$\Delta\varphi = I\mathcal{R}$	$\Delta\varphi = -L\dfrac{dI}{dt}$
capacitance C	resistance $\mathcal{R}$	inductance L
unit$[C] =$ F	unit$[\mathcal{R}] = \Omega$	unit$[L] =$ H

Table 13.1 summarizes the "lumped" circuit elements: capacitor, resistor, and inductor. If capacitors or inductors are nearby one another in a circuit, then **mutual** capacitances and inductances become significant and depend strongly on the relative geometry of the circuit elements.

13.3 RC Circuits

Consider charging a capacitor using the RC circuit of Fig. 13.3. How does the current vary with time?

Kirchoff's rules still apply to time varying currents so long as the currents don't change too rapidly. Currents and voltages must not change much in the time it takes light to travel around the circuit, else at very high frequencies, the circuit acts like an antenna or transmitter with electromagnetic radiation carrying energy away and charge bunching up and spreading out along the wires.

Kirchoff's loop rule implies that the sum of the voltages

$$+\Delta\varphi - \frac{Q}{C} - I\mathcal{R} = 0 \tag{13.9}$$

or

$$\Delta\varphi - \frac{1}{C}Q - \mathcal{R}\frac{dQ}{dt} = 0. \tag{13.10}$$

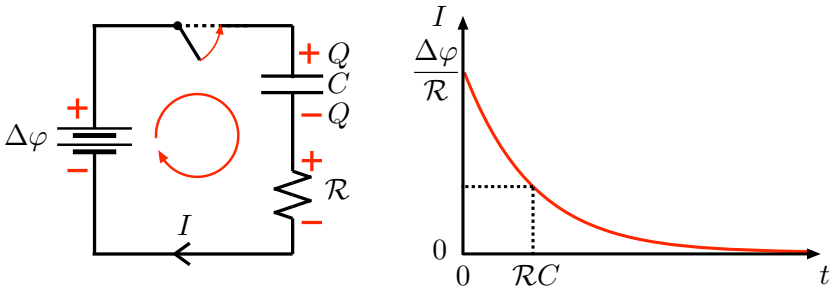


Figure 13.3: Closing the switch charges the capacitor with an exponentially decreasing current until the voltage across it equals that of the battery.

This is a **differential equation** for the charge $Q[t]$ as a function of time t. Separate the dependent and independent variables on opposites sides of the equation by writing

$$0 < C\Delta\varphi - Q = \mathcal{R}C\frac{dQ}{dt} \tag{13.11}$$

and

$$\frac{dQ}{C\Delta\varphi - Q} = \frac{dt}{\mathcal{R}C}. \tag{13.12}$$

Integrate both sides to get

$$\int_0^Q \frac{dQ'}{C\Delta\varphi - Q'} = \int_0^t \frac{dt'}{\mathcal{R}C}. \tag{13.13}$$

and

$$-\log[C\Delta\varphi - Q']\Big|_0^Q = \frac{t'}{\mathcal{R}C}\Big|_0^t \tag{13.14}$$

and

$$\log[C\Delta\varphi - Q] - \log[C\Delta\varphi] = \log\left[1 - \frac{Q}{C\Delta\varphi}\right] = -\frac{t}{\mathcal{R}C}. \tag{13.15}$$

Exponentiate both sides to find the charge

$$Q[t] = C\Delta\varphi\left(1 - e^{-t/\mathcal{R}C}\right). \tag{13.16}$$

and the current

$$I[t] = \frac{dQ[t]}{dt} = \frac{\Delta\varphi}{\mathcal{R}}e^{-t/"C} = I[0]e^{-t/t_C}, \tag{13.17}$$

where $t_C = RC$ is the **relaxation time** or **time constant** of the RC circuit. As checks, note that $I[0] = \Delta\varphi/\mathcal{R}$ and $I[\infty] = 0$.

13.4 RL Circuits

Consider the RL circuit of Fig. 13.4. How does the current vary with time?

Kirchoff's loop rule implies that the sum of the voltages

$$+\Delta\varphi - L\frac{dI}{dt} - I\mathcal{R} = 0. \tag{13.18}$$

This RL differential equation is formally the same as the Eq. 13.10 RC differential equation with the substitutions $Q \to I$, $\mathcal{R} \to L$, $C \to 1/R$. Hence its solution is

$$I[t] = \frac{\Delta\varphi}{\mathcal{R}}\left(1 - e^{-t/(L/\mathcal{R})}\right) \tag{13.19}$$

or

$$I[t] = I[\infty]\left(1 - e^{-t/t_L}\right), \tag{13.20}$$

where $t_L = L/\mathcal{R}$ is the **inductive time constant** of the RL circuit. As checks, note that $I[0] = 0$ and $I[\infty] = \Delta\varphi/\mathcal{R}$.

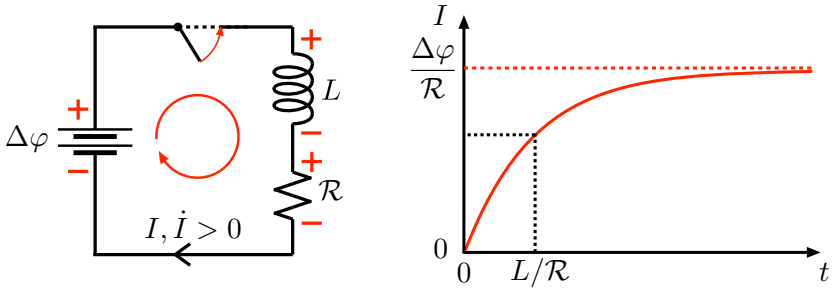


Figure 13.4: Closing the switch induces a counter-voltage across the inductor that causes onential rise in the current to its inductor-less value.

13.5 LC Circuits

There is a powerful analogy between mechanical and electronic oscillators. The Fig. 13.5 linear spring of stiffness κ attached to a block of mass m obeys Newton's law

$$m\ddot{x} = ma_x = F_x = -kx \tag{13.21}$$

or

$$m\ddot{x} + kx = 0, \tag{13.22}$$

which implies cosinusoidal displacement

$$x = x_M \cos[\omega_0 t + \phi] \tag{13.23}$$

and sinusoidal velocity

$$v = \dot{x} = -\omega_0 x_0 \sin[\omega_0 t + \phi] = -v_M \sin[\omega_0 t + \phi] \tag{13.24}$$

at the natural frequency

$$\omega_0 = \sqrt{\frac{\kappa}{m}}. \tag{13.25}$$

Its energy naturally oscillates between kinetic and potential forms preserving the total energy

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}\kappa x^2 = \frac{1}{2}mv_M^2 = \frac{1}{2}\kappa x_M^2. \tag{13.26}$$

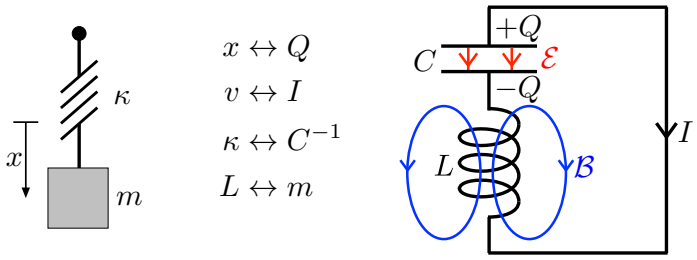


Figure 13.5: New physics, old math: a simple harmonic oscillator (left) and its electronic analogue LC circuit (right).

Similarly, the Fig. 13.5 capacitor of capacitance C wired to an inductor of inductance L obeys Kirchoff's loop rule

$$-L\dot{I} - \frac{Q}{C} = 0 \tag{13.27}$$

or

$$L\ddot{Q} + \frac{1}{C}Q = 0, \tag{13.28}$$

which implies cosinusoidal charging

$$Q = Q_M \cos[\omega_0 t + \phi], \tag{13.29}$$

and sinusoidal currents

$$I = \dot{Q} = -\omega_0 Q_M \sin[\omega_0 t + \phi] = -I_M \sin[\omega_0 t + \phi] \tag{13.30}$$

at the natural frequency

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{L/\mathcal{R}}\frac{1}{\mathcal{R}C}} = \frac{1}{\sqrt{t_L t_C}} \tag{13.31}$$

and natural period

$$t_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{t_L t_C}. \tag{13.32}$$

Its energy naturally oscillates between magnetic and electric forms preserving the total energy

$$E = E_{\mathcal{B}} + E_{\mathcal{E}} = \frac{1}{2}L\dot{I}^2 + \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}L\dot{I}_M^2 = \frac{1}{2}\frac{Q_M^2}{C}. \tag{13.33}$$

The inductance and capacitance of the LC circuit are analogous to the mass of the block and the inverse stiffness of the spring. The inductor resists changes in current like the mass resists changes in velocity. The capacitor increases or decreases charge like the spring stretches or compresses displacement.

13.6 RLC Circuits

The Fig. 13.6 linear spring of stiffness κ attached to a block of mass m and a linear damper of viscosity γ obeys Newton's law

$$m\ddot{x} = ma_x = F_x = -kx - \gamma v_x \tag{13.34}$$

or

$$m\ddot{x} + \gamma\dot{x} + kx = 0, \tag{13.35}$$

which implies exponentially decaying cosinusoidal displacement

$$x = x_M e^{-\gamma t/2m}\cos[\omega_\gamma t + \phi] \tag{13.36}$$

at the reduced squared frequency

$$\omega_\gamma^2 = \omega_0^2 - \left(\frac{\gamma}{2m}\right)^2. \tag{13.37}$$

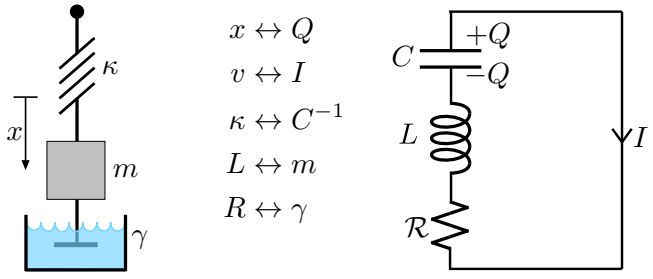


Figure 13.6: New physics, old math: a damped harmonic oscillator (left) and its electronic analogue RLC circuit (right).

Similarly, the Fig. 13.5 capacitor of capacitance C wired to an inductor of inductance L and a resistor of resistance R obeys Kirchoff's loop rule

$$-L\dot{I} - \frac{Q}{C} - I\mathcal{R} = 0 \tag{13.38}$$

or

$$L\ddot{Q} + \mathcal{R}\dot{Q} + \frac{1}{C}Q = 0, \tag{13.39}$$

which implies exponentially decaying cosinusoidal charging

$$Q = Q_M e^{-Rt/2L}\cos[\omega_\gamma t + \phi] \tag{13.40}$$

at the reduced squared frequency

$$\omega_\gamma^2 = \omega_0^2 - \left(\frac{\mathcal{R}}{2L}\right)^2. \tag{13.41}$$

13.7 Resonance

Finally, when sinusoidally forced, the damped harmonic oscillator and the RLC circuit respond **resonantly** when the forcing frequency ω equals the natural frequency ω_0, as in Fig. 13.7.

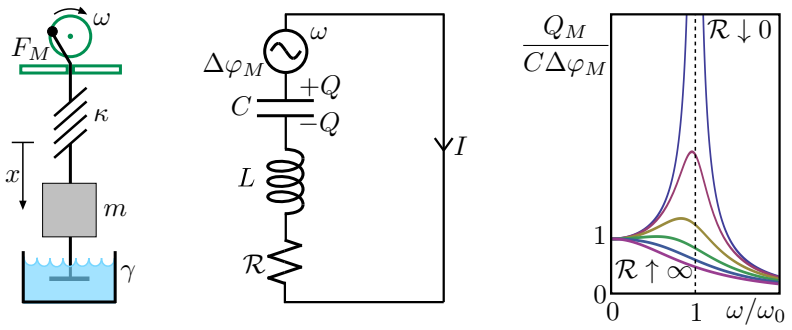
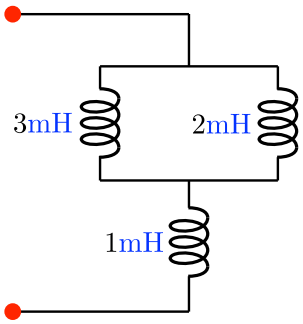


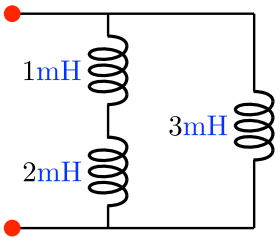
Figure 13.7: New physics, old math: a forced damped harmonic oscillator (left), its electronic analogue AC RLC circuit (center), and its resonant frequency response (right).

Resonance enables RLC circuits to selectively enhance incoming radio signals. Rotating a dial changes the gap d between the capacitor plates, which changes the capacitance C, which changes the resonance frequency ω_0, which produces an enhanced response to a radio station's carrier frequency ω when the RLC circuit is inductively coupled to an antenna.

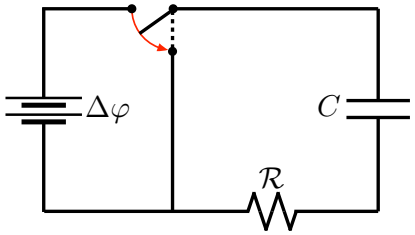
13.8 Problems

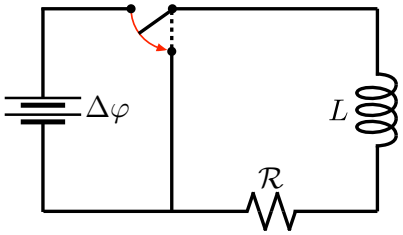
1. What is the equivalent inductance of these series-parallel combinations of inductors, neglecting any mutual inductance?





2. Use Kirchoff's rules to find differential equations for charge or current in the circuits below as the currents decay *after* the switches are thrown to disconnect from the battery. Separate, integrate, and solve the differential equations for current as a function of time.





3. Show that the Eq. 13.36 exponentially decaying cosinusoidal displacement at the Eq. 13.37 reduced squared frequency solves the Eq. 13.35 damped oscillator law of motion.

Appendix A

Electromagnetic Units

Although best for practical applications, SI units obscure the symmetry and elegance of electromagnetism and are not used in advanced texts. Table A.1 summarizes familiar electromagnetic equations in three commonly used systems of units.

Table A.1: Key electromagnetic equations in three common systems of units.

Natural	Gaussian	SI
$\mathcal{E} = \dfrac{Q}{4\pi r^2}$	$\mathcal{E} = \dfrac{Q}{r^2}$	$\epsilon_0 \mathcal{E} = \dfrac{Q}{4\pi r^2}$
$\mathcal{B} = \dfrac{I}{2\pi s}$	$\mathcal{B} = \dfrac{2}{c}\dfrac{I}{s}$	$\mu_0^{-1} \mathcal{B} = \dfrac{I}{2\pi s}$
$\vec{F} = q\left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right)$	$\vec{F} = q\left(\vec{\mathcal{E}} + \dfrac{\vec{v}}{c} \times \vec{\mathcal{B}}\right)$	$\vec{F} = q\left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right)$
$\Phi_{\mathcal{E}} = Q$	$\Phi_{\mathcal{E}} = 4\pi Q$	$\epsilon_0 \Phi_{\mathcal{E}} = Q$
$\Phi_{\mathcal{B}} = 0$	$\Phi_{\mathcal{B}} = 0$	$\Phi_{\mathcal{B}} = 0$
$\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}$	$\Gamma_{\mathcal{E}} = -\dfrac{1}{c}\dot{\Phi}_{\mathcal{B}}$	$\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}$
$\Gamma_{\mathcal{B}} = +\dot{\Phi}_{\mathcal{E}} + I$	$\Gamma_{\mathcal{B}} = +\dfrac{1}{c}\dot{\Phi}_{\mathcal{E}} + \dfrac{1}{c} I$	$\mu_0^{-1}\Gamma_{\mathcal{B}} = \epsilon_0 \dot{\Phi}_{\mathcal{E}} + I$

Appendix B

Dot & Cross Products

The **scalar** or **dot** product and the **vector** or **cross** product are of complementary definition and utility. If $\vec{a}$ and $\vec{b}$ are two vectors with magnitudes a and b, then their dot product

$$\vec{a} \cdot \vec{b} = ab \cos \alpha = \vec{b} \cdot \vec{a} \tag{B.1}$$

where α is the angle between the vectors, and their cross product

$$\vec{a} \times \vec{b} = ab \sin \alpha \, \hat{u} = -\vec{b} \times \vec{a}, \tag{B.2}$$

where $\hat{u}$ is a unit vector perpendicular to both factors and pointing in the direction a right-handed screw would advance when rotated from $\vec{a}$ to $\vec{b}$.

If $\{\hat{x}, \hat{y}, \hat{z}\}$ are a right-handed orthonormal basis set, then

$$\begin{aligned} \hat{x} \cdot \hat{x} &= 1, \quad \hat{y} \cdot \hat{x} = 0, \quad \hat{z} \cdot \hat{x} = 0, \\ \hat{x} \cdot \hat{y} &= 0, \quad \hat{y} \cdot \hat{y} = 1, \quad \hat{z} \cdot \hat{y} = 0, \\ \hat{x} \cdot \hat{z} &= 0, \quad \hat{y} \cdot \hat{z} = 0, \quad \hat{z} \cdot \hat{z} = 1, \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} \hat{x} \times \hat{x} &= \vec{0}, \quad \hat{y} \times \hat{x} = -\hat{z}, \quad \hat{z} \times \hat{x} = +\hat{y}, \\ \hat{x} \times \hat{y} &= +\hat{z}, \quad \hat{y} \times \hat{y} = \vec{0}, \quad \hat{z} \times \hat{y} = -\hat{x}, \\ \hat{x} \times \hat{z} &= -\hat{y}, \quad \hat{y} \times \hat{z} = +\hat{x}, \quad \hat{z} \times \hat{z} = \vec{0}. \end{aligned} \tag{B.4}$$

The dot product is unity where the cross product is zero and zero where the

cross product is unit vectors. Consequently, for generic vectors, the dot product

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (\hat{x}a_x + \hat{y}a_y + \hat{z}a_z)\cdot(\hat{x}b_x + \hat{y}b_y + \hat{z}b_z)\\ &= \hat{x}\cdot\hat{x}\,a_xb_x + \hat{x}\cdot\hat{y}\,a_xb_y + \hat{x}\cdot\hat{z}\,a_xb_z\\ &+ \hat{y}\cdot\hat{x}\,a_yb_x + \hat{y}\cdot\hat{y}\,a_yb_y + \hat{y}\cdot\hat{z}\,a_yb_z\\ &+ \hat{z}\cdot\hat{x}\,a_zb_x + \hat{z}\cdot\hat{y}\,a_zb_y + \hat{z}\cdot\hat{z}\,a_zb_z\\ &= a_xb_x + 0 + 0\\ &+ 0 + a_yb_y + 0\\ &+ 0 + 0 + a_zb_z \end{aligned} \tag{B.5}$$

$$\vec{a}\cdot\vec{b} = a_xb_x + a_yb_y + a_zb_z \tag{B.6}$$

and the cross product

$$\begin{aligned}\vec{a}\times\vec{b} &= (\hat{x}a_x + \hat{y}a_y + \hat{z}a_z)\times(\hat{x}b_x + \hat{y}b_y + \hat{z}b_z)\\ &= \hat{x}\times\hat{x}\,a_xb_x + \hat{x}\times\hat{y}\,a_xb_y + \hat{x}\times\hat{z}\,a_xb_z\\ &+ \hat{y}\times\hat{x}\,a_yb_x + \hat{y}\times\hat{y}\,a_yb_y + \hat{y}\times\hat{z}\,a_yb_z\\ &+ \hat{z}\times\hat{x}\,a_zb_x + \hat{z}\times\hat{y}\,a_zb_y + \hat{z}\times\hat{z}\,a_zb_z\\ &= \vec{0} + \hat{z}\,a_xb_y - \hat{y}\,a_xb_z\\ &- \hat{z}\,a_yb_x + \vec{0} + \hat{x}\,a_yb_z\\ &+ \hat{y}\,a_zb_x - \hat{x}\,a_zb_y + \vec{0} \end{aligned} \tag{B.7}$$

$$\vec{a}\times\vec{b} = \hat{x}(a_yb_z - a_zb_y) + \hat{y}(a_zb_x - a_xb_z) + \hat{z}(a_xb_y - a_yb_x). \tag{B.8}$$

Appendix C

Coordinate Systems

Multiple coordinate systems are useful in electromagnetism to solve problems of different symmetries, including rectangular, spherical, and cylindrical.

C.1 Curvilinear Coordinates

Consider a general curvilinear coordinate system $\{u_1, u_2, u_3\}$ whose axes are orthogonal at point. An infinitesimally small cube with edges parallel to the local curvilinear coordinate directions has edges of lengths $h_1 du_1$, $h_2 du_2$, and $h_2 du_2$, as in Fig. C.1.

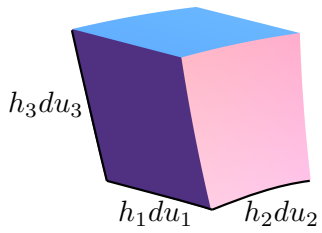


Figure C.1: Generic coordinate system $\{u_1, u_2, u_3\}$ and infinitesimal volume element of size $h_1 du_1$ by $h_2 du_2$ by $h_3 du_3$.

The square of the distance across opposite corners of the cube is

$$ds^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2. \qquad \text{(C.1)}$$

The volume of the cube is

$$dV = (h_1 du_1)(h_2 du_2)(h_3 du_3) = h_1 h_2 h_3 \, du_1 du_2 du_3. \qquad \text{(C.2)}$$

C.2 Polar Spherical Coordinates

Define spherical coordinates $\{u_1, u_2, u_3\} = \{r, \theta, \phi\}$ by

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \end{aligned} \tag{C.3}$$

where r is the radial distance form the origin, θ is the co-latitude and ϕ is the longitude, as in Fig. C.2. By inspection, the scale factors

$$\begin{aligned} h_1 &= 1, \\ h_2 &= r, \\ h_3 &= r\sin\theta. \end{aligned} \tag{C.4}$$

Hence, the diagonal square distance

$$ds^2 = dr^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2, \tag{C.5}$$

and the elemental volume

$$dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi. \tag{C.6}$$

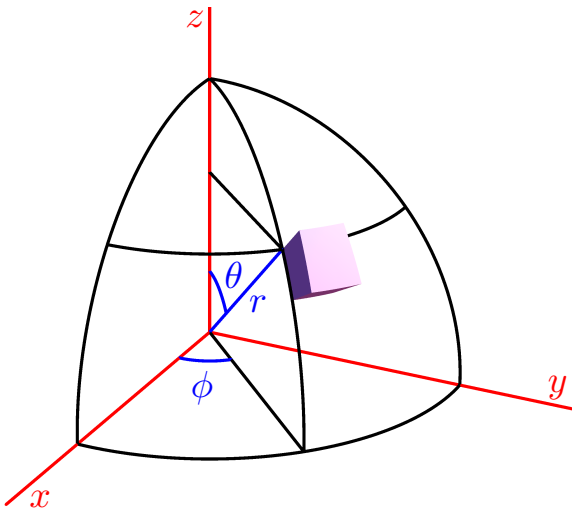


Figure C.2: Polar spherical coordinate system $\{r, \theta, \phi\}$ and infinitesimal volume element of size dr by $r\,d\theta$ by $r\sin\theta\,d\phi$.

C.3 Cylindrical Coordinates

Define cylindrical coordinates $\{u_1, u_2, u_3\} = \{s, \phi, z\}$ by

$$\begin{aligned} x &= s\cos\phi, \\ y &= s\sin\phi, \\ z &= z, \end{aligned} \tag{C.7}$$

where $s = r_\perp$ is the perpendicular distance (or "separation") from the z-axis and ϕ is the longitude, as in Fig. C.3. By inspection, the scale factors

$$\begin{aligned} h_1 &= 1, \\ h_2 &= s, \\ h_3 &= 1. \end{aligned} \tag{C.8}$$

Hence, the diagonal square distance

$$ds^2 = dr^2 + (s\,d\phi)^2 + dz^2 = dr^2 + s^2 d\phi^2 + dz^2 \tag{C.9}$$

and the elemental volume

$$dV = (ds)(s\,d\phi)(dz) = s\,ds d\phi\,dz. \tag{C.10}$$

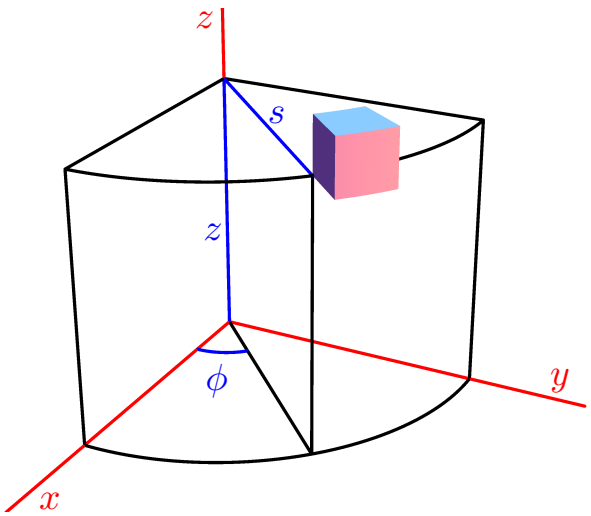


Figure C.3: Cylindrical coordinate system $\{s, \phi, z\}$ and infinitesimal volume element of size ds by $s\,d\phi$ by dz.

Appendix D

Spherical Geometry

In working electromagnetic problems with spherical symmetry, it is important to remember the geometry of a sphere, as in Fig. D.1. Using the polar spherical coordinates of Appendix C, its circumference

$$C = \oint_{r=R} d\ell = \int_0^{2\pi} R d\phi = 2\pi R. \tag{D.1}$$

Its equatorial area

$$A = \iint_{r<R} da = \int_{r=0}^{R} \int_{\phi=0}^{2\pi} (r d\phi) dr = \pi R^2. \tag{D.2}$$

Its surface area

$$S = \oiint_{r=R} da = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (R \sin\theta \, d\phi)(R d\theta) = 4\pi R^2. \tag{D.3}$$

Its volume

$$V = \iiint_{r<R} dV = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (r \sin\theta \, d\phi)(r d\theta) = \frac{4}{3}\pi R^3. \tag{D.4}$$

These geometric quantities are related in pairs. A differential equatorial area is the radial extrusion of the circumference, $dA = C dR$ or

$$C = \frac{dA}{dR}. \tag{D.5}$$

A differential volume is the radial extrusion of the surface area, $dV = S dR$ or

$$S = \frac{dV}{dR}. \tag{D.6}$$

In addition, the surface area is quadruple the equatorial area,

$$S = 4A. \tag{D.7}$$

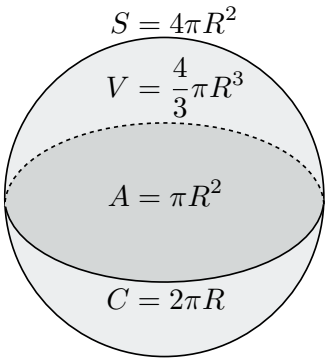


Figure D.1: Similar but distinct formulas for the CAVS – circumference C, equatorial area A, volume V, and surface area S – of a sphere of radius R.

The circumference has SI unit of **meter**,

$$\text{unit}[C] = \text{m}, \tag{D.8}$$

while the equatorial and surface areas have SI unit of **square meter**,

$$\text{unit}[A] = \text{unit}[S] = \text{m}^2, \tag{D.9}$$

and the volume has SI unit of **cubic meter**,

$$\text{unit}[V] = \text{m}^3. \tag{D.10}$$

Appendix E

Function Notation

Standard mathematics notation suffers from a serious ambiguity involving parentheses. In particular, parentheses can be used to denote multiplication, as in $a(b+c) = ab+ac$ and $f(g) = fg$, or they can be used to denote functions evaluated at arguments, as in $f(t)$ and $g(b+c)$. It can be a struggle to determine the intended meaning from context.

To avoid ambiguity, this text always uses round parentheses $(\bullet)$ to group for multiplication and square brackets $[\bullet]$ to list function arguments. Thus, $a(b) = ab$ denotes the product of two factors a and b, while $f[x]$ denotes a function f evaluated at an argument x. *Mathematica* [2] employs the same convention.

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[1] Isaac Newton, *Philosophiæ Naturalis Principia Mathematica (Mathematical Principles of Natural Philosophy)*, London (1687).

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[4] Richard P. Feynman, Robert B. Leighton, Matthew Sands, *The Feynman Lectures on Physics* (Addison-Wesley, 1963), Volume II, page 1-6.

[5] Bob Shaw, "Light of other days", *Analog*, Volume 77, Number 6 (August, 1966).